3-Manifolds Exercises

Sheet 1

Exercise 1.
A knot diagram D_K of a knot K is called **3-colorable** if one can color each arc in exactly one of three colors such that we use every color and at each crossing all three colors or only on color meet.

(a) Show that 3-colorability is a property of the knot K.

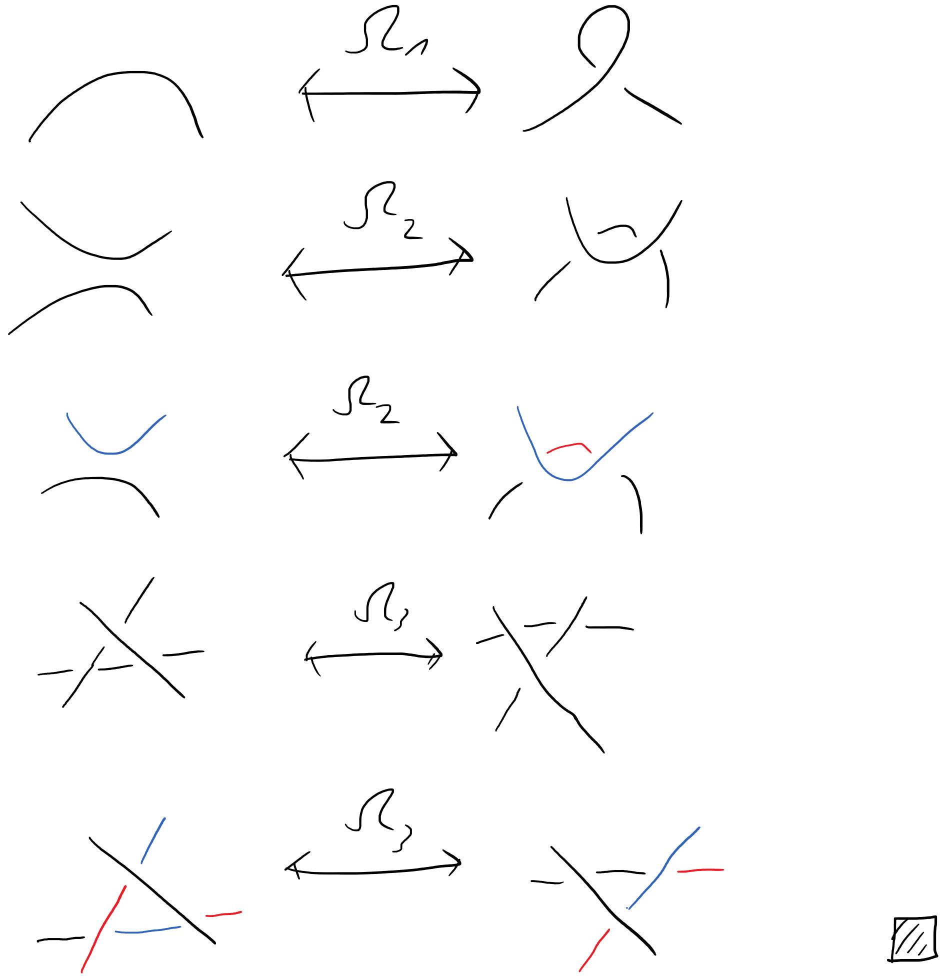
(b) Deduce that the trefoil is non-trivial (i.e. not isotopic to the unknot).

(c) Which other knots can you distinguish from each other via 3-colorability?

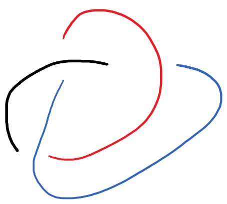
(a) CLAIM: 3-COL. is a prop of a knot K

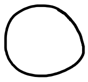
i.e. 3-COL. is invariant under $\Omega_1 / \Omega_2 / \Omega_3$

PROOF:



(b)

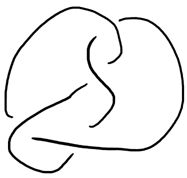
 $\Rightarrow$ trefoil is 3-colorable

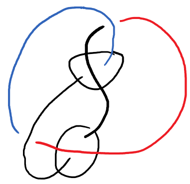
 $\Rightarrow$ unknot is <u>NOT</u> 3-colorable

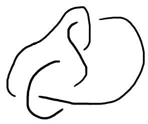
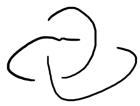
(a)

$\Rightarrow$  $\neq$ 

(c)



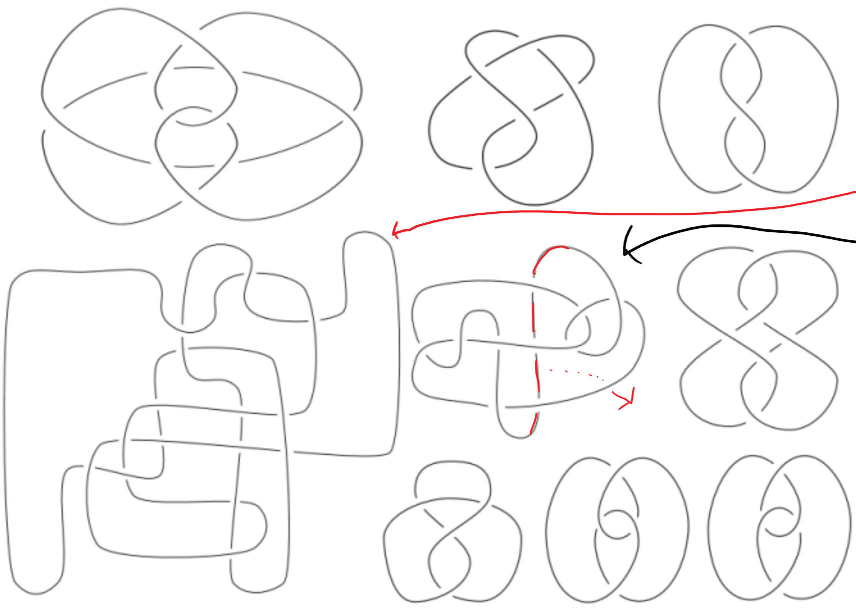
 $\Rightarrow$ is <u>NOT</u> 3-colorable

$\Rightarrow$  $\neq$ 

Exercise 2.

Determine the isotopy type of the following knots and links.

Hint: The diagram in in the middle is called culprit. The reason is that you first have to make the diagram more complicated (in therms of number of crossing) before you can simplify it. The diagram on the lower left is called Thistlethwaite knot. For many people it turned out to be complicated to determine its isotopy type.



UNKNOT

UNKNOT

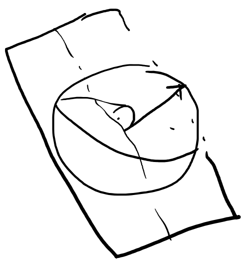
Exercise 3.

(a) Any knot admits a regular projection (i.e. prove Lemma 1.2).
Bonus: Show that a generic projection of a given knot is regular.
Hint: First, you should make the word 'generic' precise.

(b) Two knot diagrams D_K and $D_{K'}$ represent isotopic knots K and K' if and only if D_K can be transformed into $D_{K'}$ via a finite sequence of Reidemeister moves and planar isotopies (i.e. prove Theorem 1.3).

(a) $\{$ orth. projections $\mathbb{R}^3 \to \mathbb{R}^2 \}$ $\xleftrightarrow{1:1}$ S^2

proj. along v $\longleftarrow$ unit vector v

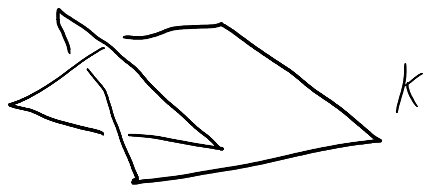


Given: $K \subset \mathbb{R}^3$

CLAIM: $\{$regular proj of $K\} \subset \{$orth. proj$\} \equiv S^2$

is open & dense

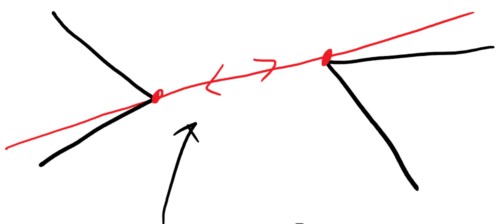
Proof sketch: Let K be a PL knot



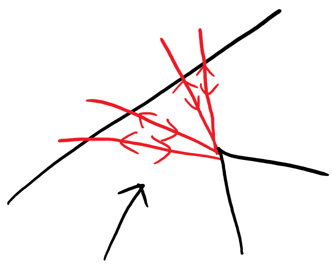
$\{$non-reg proj of $K\}$ = finite union of pts & curves on S^2

$\subset \mathbb{R}^3$

(iii)

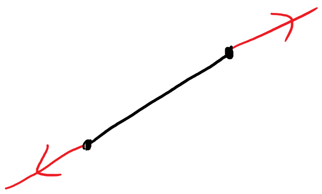


2 pts on S^2



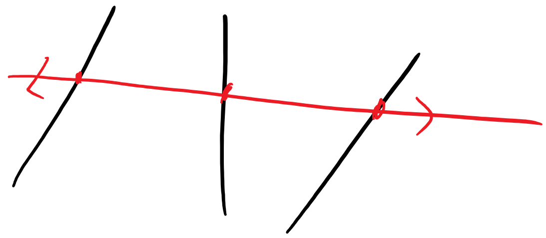
2 arcs on S^2

(i)



$\longrightarrow$ 2 pts on S^2

(ii)



$\longrightarrow$ 2 curves on S^2

$\blacksquare$

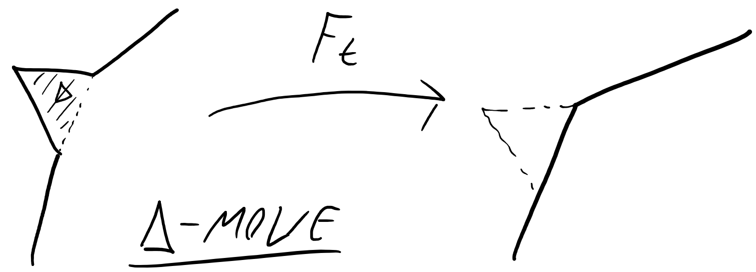
(b) CLAIM: $K_0 \sim K_1 \iff D_{K_0} \longrightarrow D_{K_1}$ via finitely many Reidemeister moves & planar isotopies

Proof: Let K_0, K_1 be PL knots

& $F: S^1 \times \underset{\substack{\Psi \\ t}}{I} \longrightarrow \mathbb{R}^3$ be a PL isotopy from K_0 to K_1

i.e. F_t is a PL knot $\forall\, t \in I$

& $F_0 = K_0$ & $F_1 = K_1$

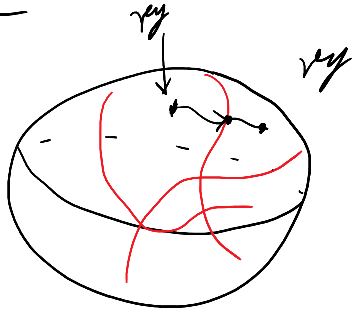


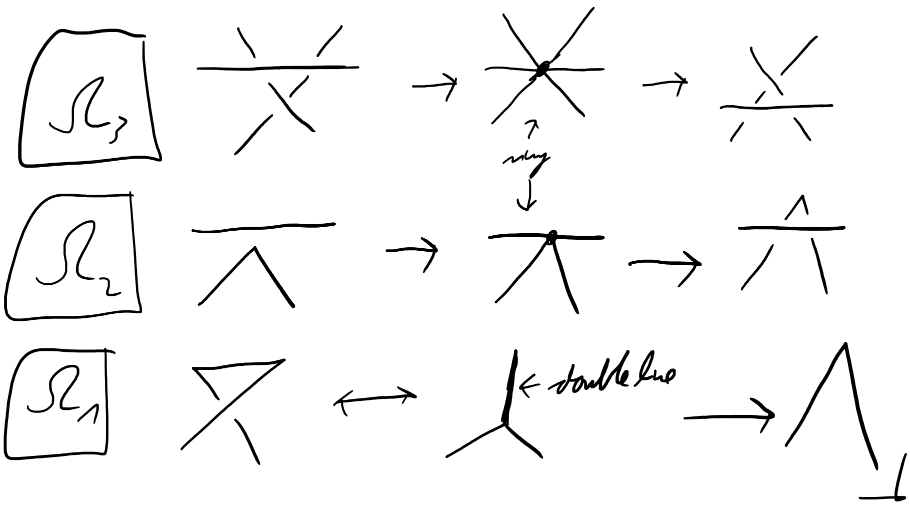
Δ-MOVE

$\Rightarrow$ Every PL-isotopy can be decomposed into Δ-moves

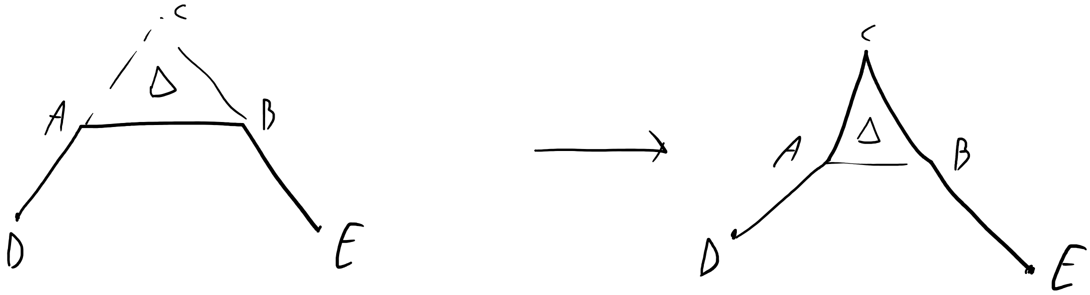
„$\Leftarrow$" Ω_i can be expressed by Δ-moves.

„$\Rightarrow$" ① Any two reg. proj of the same knot K are related by Ω_i-moves or planar isotopies



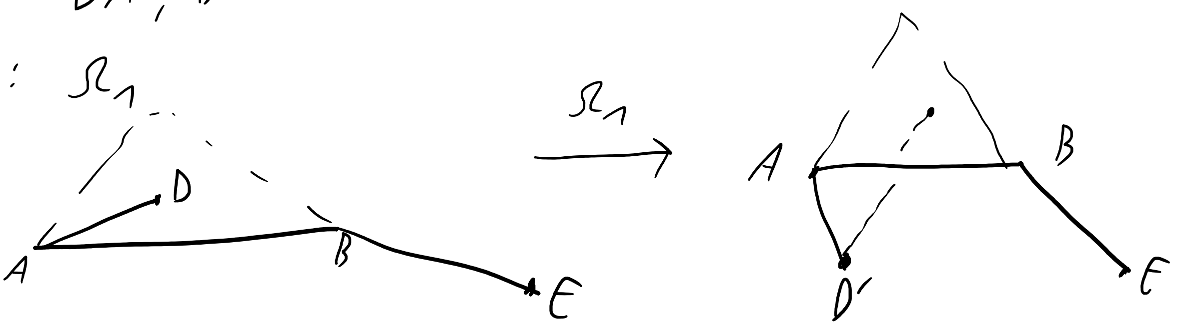


(2)



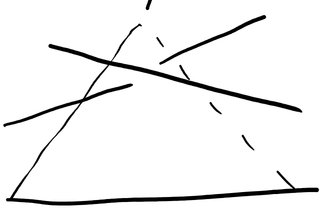
(a) OE $DA, BE \cap \Delta = \emptyset$

else: Ω_1

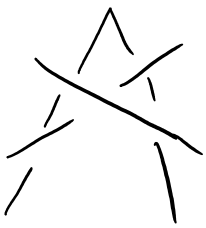


(b) Decompose Δ into smaller triangles s.t.

(i)



$\longrightarrow$ $\Delta = \Omega_3, \Omega_2$



(ii)

$\longrightarrow$ Δ = planar isotopy or Ω_2

(iii)

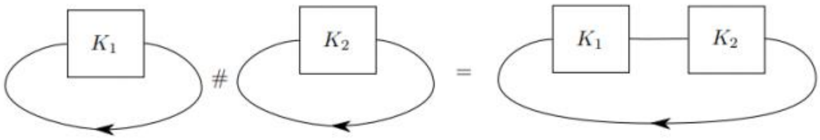
$\longrightarrow$ $\Delta = \Omega_2$ or planar

(iv)

$\longrightarrow$ planar

∎

Exercise 4.
The **connected sum** of two *oriented* knots K_1 and K_2 is defined in the following picture.



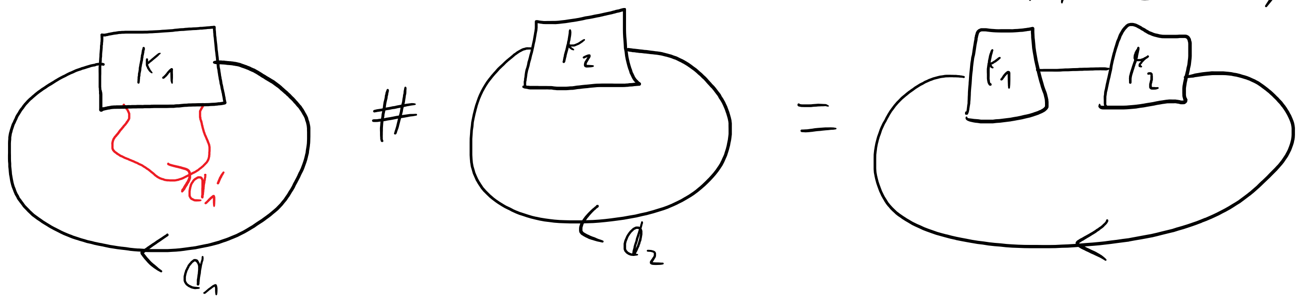
(a) Show that the connected sum is well-defined. Given an example showing that this is not true anymore if we work with unoriented knots.

(b) $K_1\#K_2$ is isotopic to $K_2\#K_1$.

(c) For which knots K_1 and K_2 is $K_1\#K_2$ isotopic to the unknot?

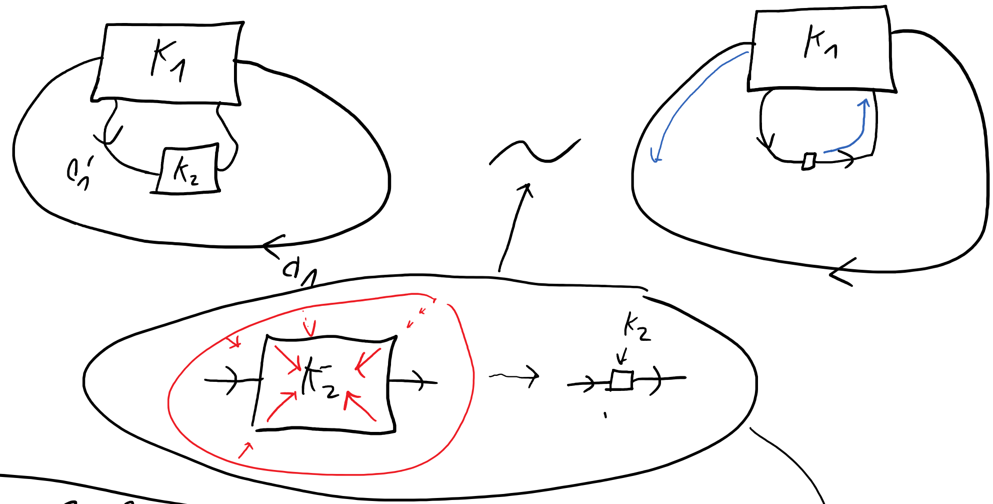
$K_1\#K_2(a_1,a_2)$

(a)

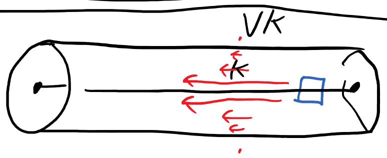


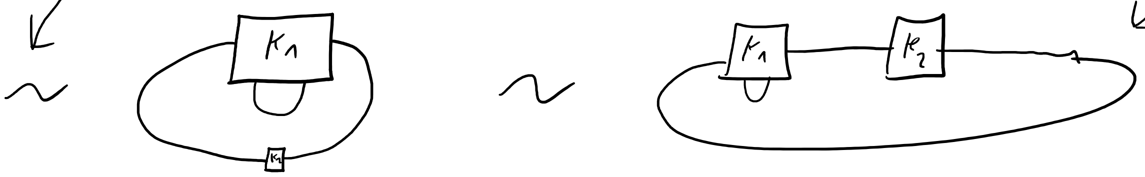
$$K_1\#K_2(a_1',a_2) \sim K_1\#K_2(a_1,a_2)$$

$K_1\#K_2(a_1',a_2) =$



$K\subset\mathbb{R}^3$ has a nbhd $VK \cong S^1\times D^2$
s.t. $K \cong S^1\times\{0\}$





$= K_1\#K_2(a_1,a_2)$

∎

(b) the same

for <u>UNORIENTED KNOTS</u>:

Let K_1 & K_2 be oriented knots s.t.

$$K_1 \not\sim -K_1 \quad \& \quad K_2 \not\sim -K_2$$

(<u>NON-INVERTIBLE</u>)

$$K_1 \# K_2, \quad K_1 \# (-K_2), \quad (-K_1) \# K_2, \quad (-K_1) \# (-K_2)$$

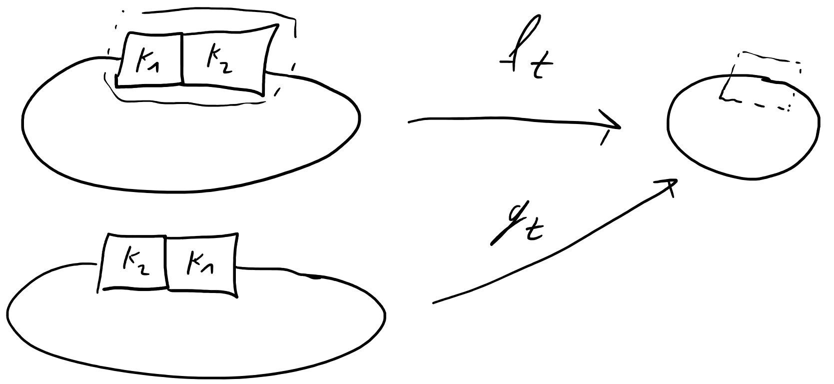
are pairwise <u>NON-ISOTOPIC</u>

(c) <u>Claim</u>: $K_1 \# K_2 \sim O \quad \Rightarrow \quad K_1 \,\&\, K_2 \sim O$

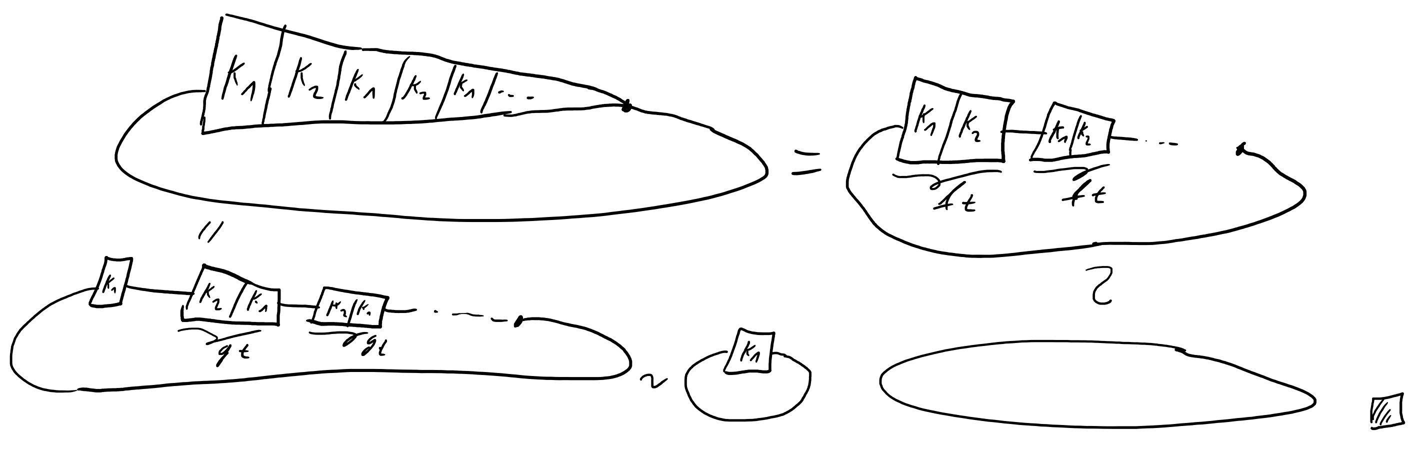
<u>Anders</u>: $K_1 \not\sim O \Rightarrow K_1 \# K_2 \not\sim O$

<u>Assume</u>: $K_1 \# K_2 \sim O$

$\Rightarrow \exists$ isotopy f_t from $K_1 \# K_2$ to O



CONSIDER THE <u>WILD</u> KNOT:



HOMOLOGY OF KNOT EXTERIORS (BONUS EXERCISE)

$$S^3 = \underbrace{VK}_{= S^1 \times D^2} \cup S^3 \setminus \mathring{V}K$$

$$VK \cap S^3 \setminus \mathring{V}K = \partial VK = S^1 \times S^1$$

$$0 = H_2(S^3) \longrightarrow \underbrace{H_1(\partial VK)}_{= \mathbb{Z}^2} \xrightarrow{\cong} \underbrace{H_1(VK)}_{= \mathbb{Z}} \oplus H_1(S^3 \setminus \mathring{V}K) \longrightarrow H_1(S^3) = 0$$

$$\Rightarrow H_1(S^3 \setminus \mathring{V}K) = \mathbb{Z}$$

$$\underbrace{H_3(S^3)}_{= \mathbb{Z}} \xrightarrow{\cong} \underbrace{H_2(\partial VK)}_{= \mathbb{Z}} \longrightarrow \underbrace{H_2(VK)}_{= 0} \oplus H_2(S^3 \setminus \mathring{V}K) \longrightarrow H_2(S^3) = 0$$

$$\Rightarrow 0$$

Topology of 3-Manifolds

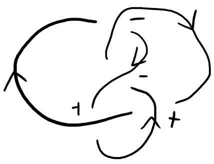
Exercise sheet 2

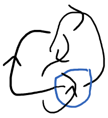
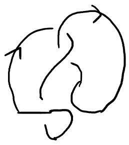
Exercise 1.
Compute the Jones polynomial of the figure eight knot in two ways:

(a) via the Kauffman polynomial, and

(b) by directly using the Skein relation.

Deduce that the figure eight knot is non-trivial.



(b) $q^{-1} V\big($  $\big) - q\,\underbrace{V(O)}_{=1} = (q^{1/2} - q^{-1/2})\, V\big($  $\big)$

L_+ $\qquad$ L_0

$V(L_0)$: neg Hopf link $= -q^{-5/2} - q^{-1/2}$

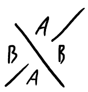
Ex 2 from lecture

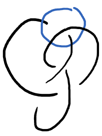
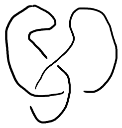
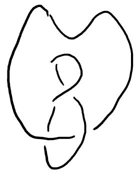
$\Rightarrow V(\text{fig } 8) = q^{-2} - q^{-1} + 1 - q + q^{2} \neq 1 = V(O)$

$\Rightarrow \text{fig } 8 \not\simeq O$

(a) $X(\text{fig } 8) = (-a)^{-3w(\text{fig}8)} \langle \text{fig } 8 \rangle = \langle \text{fig } 8 \rangle$

$w(\text{fig } 8) = 0$

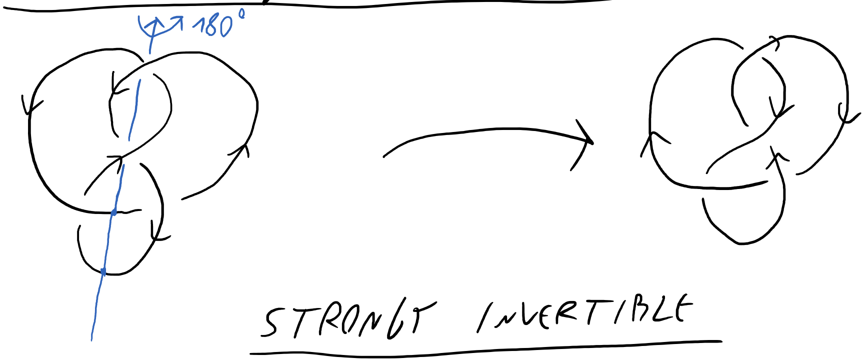
$\langle$  $\rangle = a \langle$  $\rangle + a^{-1} \langle\,)\,(\,\rangle$

$\langle$  $\rangle = a \langle$  $\rangle + a^{-1} \langle$  $\rangle$

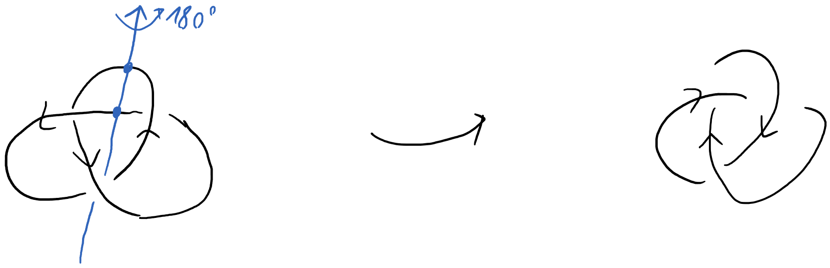
$= \ldots$

Exercise 2.
A knot K is called **amphicheiral** if it is isotopic to its mirror $\overline{K}$. An oriented knot K is called **invertible** if its is isotopic to itself with the reversed orientation $-K$.
Are the trefoil and the figure eight knot amphicheiral or invertible?

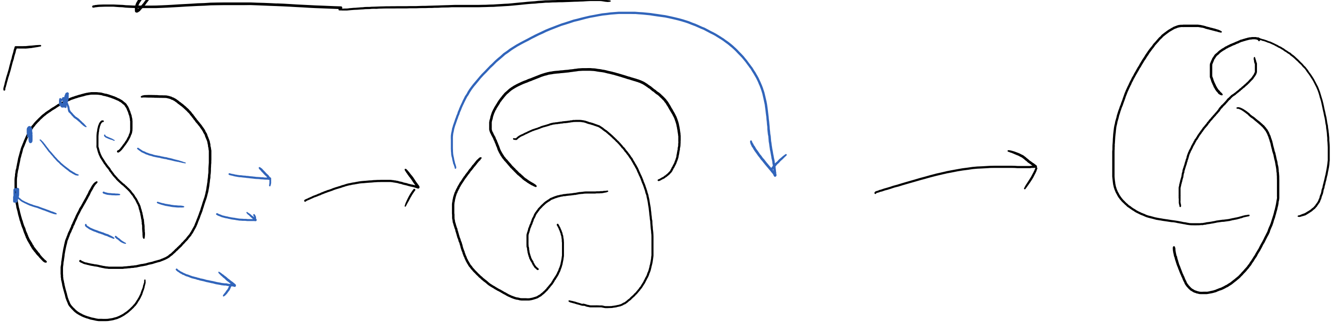
* trefoil & fig-8 are invertible:



STRONGLY INVERTIBLE



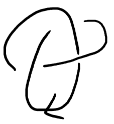
* Fig 8 is amphicheiral:



* trefoil is NOT amphicheiral (i.e. CHEIRAL)

Ex (1) from lecture: $V($  $) = q^{-1} + q^{-3} - q^{-4}$

left-handed trefoil

$V($  $) =$ compute ✕

right-handed trefoil

or better: $V(\overline{L})(q) = V(L)(q^{-1})$

Exercise 3.

Let L be an oriented link with an odd (respectively even) number of components. Then its Jones polynomial $V(L)$ consists only of terms of the form q^k (respectively $q^{k+1/2}$) for integers $k \in \mathbb{Z}$.

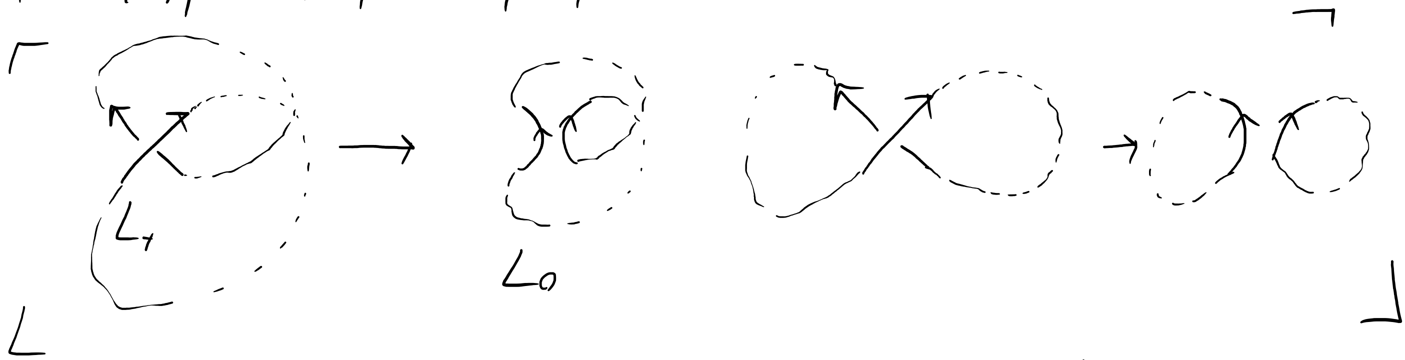
Hint: Use the skein relation and an induction argument.

<u>Proof</u>

* Ex(0) $\Rightarrow$ $V(\bigcirc \cdots \bigcirc) = \left(-q^{-1/2} - q^{1/2}\right)^{n-1} = q^{\frac{n-1}{2}}\left(-q^{-1}-1\right)^{n-1}$

$\Rightarrow$ claim is true for trivial links

* $\#|L_+| = \#|L_-| = \#|L_0| \pm 1$



* if the claim holds true for two of $V(L_+)$, $V(L_-)$, $V(L_0)$

$\Rightarrow$ it holds true for the third.

$$\left[\; q^{-1} V(L_+) - q\, V(L_-) = \left(q^{1/2} - q^{-1/2}\right) V(L_0) \;\right]$$

* the claim follows by induction L.2.6. $\blacksquare$

Exercise 4.

(a) For oriented knots K_1 and K_2 we have $V(K_1\#K_2) = V(K_1)V(K_2)$. Can you prove something similar for oriented links?

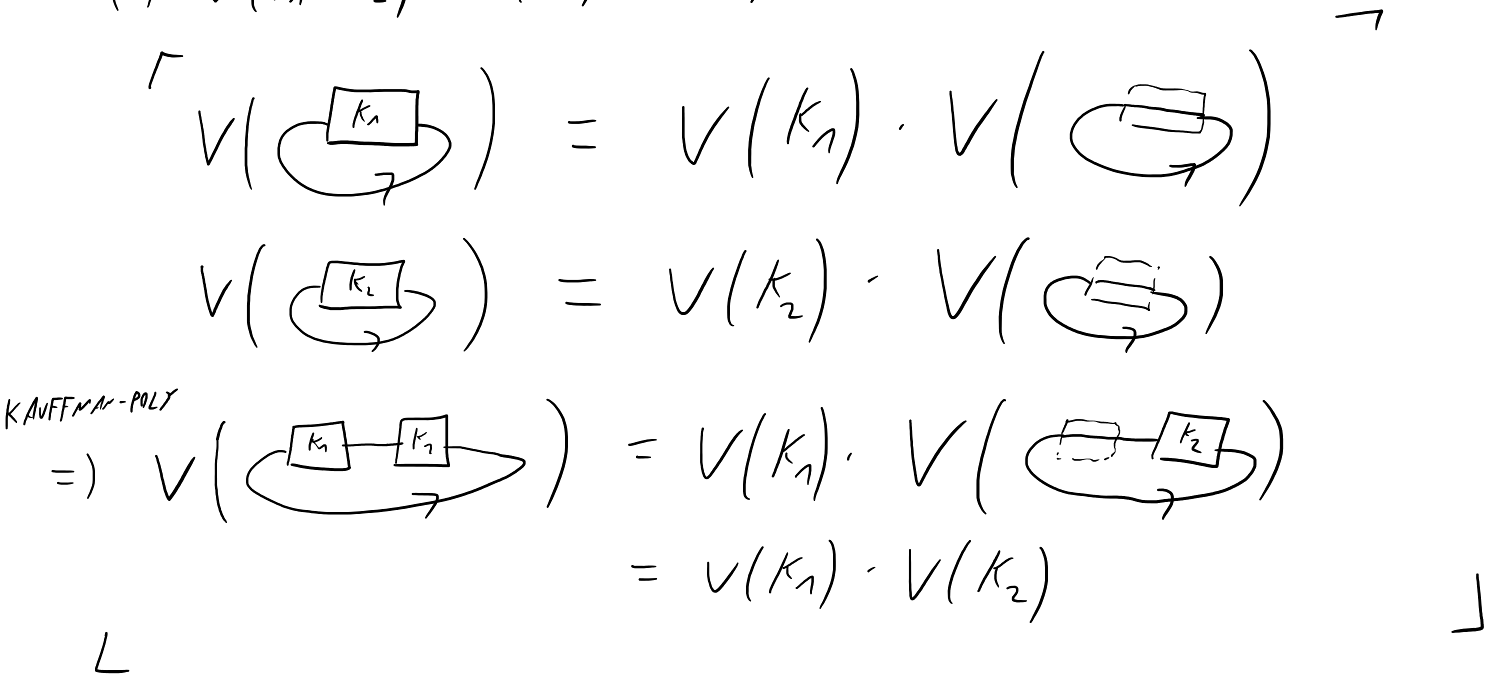
(b) For the disjoint union $L_1 \sqcup L_2$ of oriented links L_1 and L_2 we have

$$V(L_1 \sqcup L_2) = -(q^{-1/2} + q^{1/2})V(L_1)V(L_2).$$

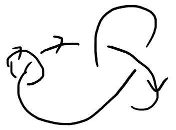
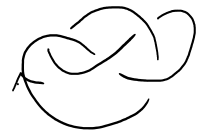
(c) Construct non-isotopic links with the same Jones polynomial.

Challenge: Can you construct non-isotopic knots with the same Jones polynomial?
Hint: The idea of the construction is similar as for links. But at the moment it will be hard to show that the constructed knots with equal Jones polynomial are really non-isotopic.

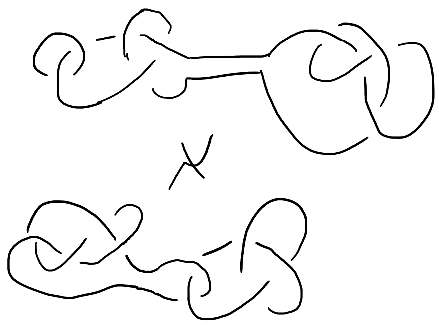
(a) $V(K_1\#K_2) = V(K_1)\cdot V(K_2)$



$L_1\#L_2$ is NOT well def for oriented links L_1, L_2

$L_1 =$  $L_2 =$ 

Two possibilities



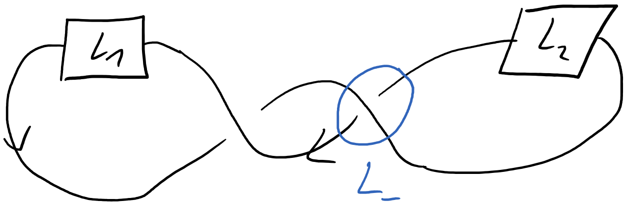
$\tilde{L}_1$ [contains an unknot]

$\tilde{L}_2$ [DOES NOT contain unknot]

but $V(L_1\#L_2) = V(L_1)\cdot V(L_2)$

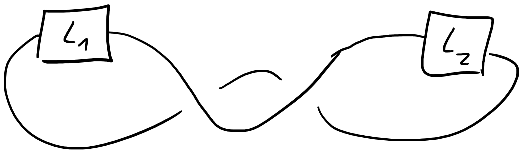
for all choices of $L_1\#L_2$ (same proof)

(b) $V(L_1 \sqcup L_2) = -(q^{-1/2}+q^{1/2})\, V(L_1)\, V(L_2)$



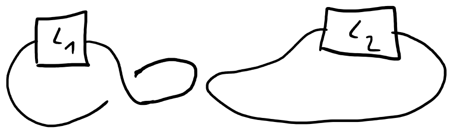
$= L_1 \# L_2$

$L_+ =$



$= L_1 \# L_2$

$L_0 =$



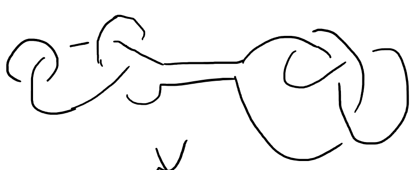
$= L_1 \sqcup L_2$

skein rel

$$\Rightarrow \quad (q^{-1}-q)\, V(L_1 \# L_2) = (q^{1/2}-q^{-1/2})\, V(L_1 \sqcup L_2)$$

$$\Rightarrow \quad V(L_1 \sqcup L_2) = \underbrace{\frac{q^{-1}-q}{q^{1/2}-q^{-1/2}}}_{= -(q^{-1/2}+q^{1/2})} \; \underbrace{V(L_1 \# L_2)}_{\overset{(a)}{=} V(L_1)\, V(L_2)}$$

(c) $\exists\ L_1, L_2$ s.t. $L_1 \not\simeq L_2$ but $V(L_1) = V(L_2)$



$=: L_1$ $\qquad L_1 = \tilde{L}_1 \# \tilde{L}_2$

$\not\simeq$

$=: L_2$ $\qquad L_2 = \tilde{L}_1 \# \tilde{L}_2$

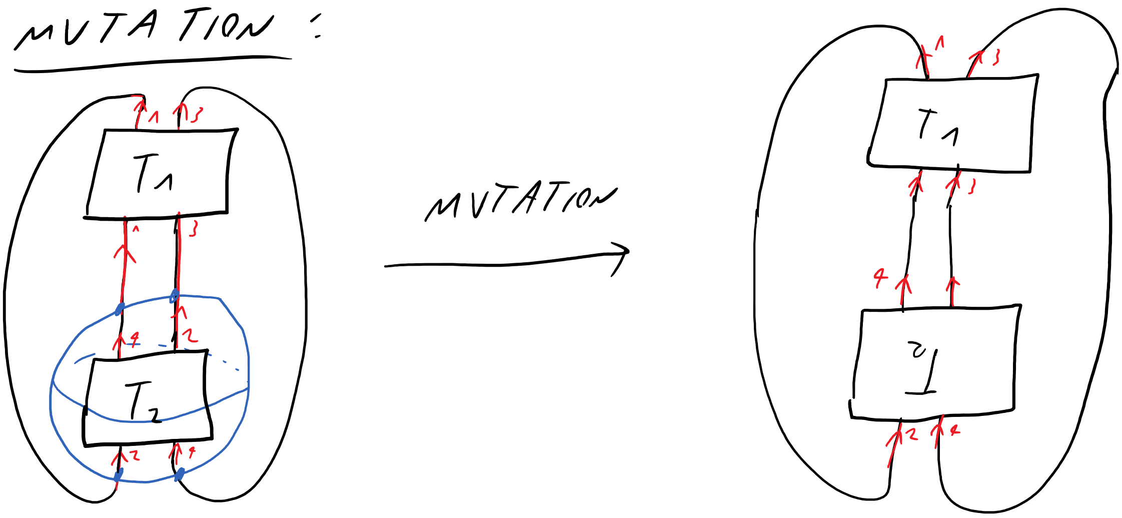
$\overset{(a)}{\Rightarrow} \quad L_1 \not\simeq L_2$ but $V(L_1) = V(L_2)$

Challenge: $\exists$ or. knots K_1, K_2 s.t. $K_1 \not\sim K_2$ but $V(K_1) = V(K_2)$

$$\boxed{\text{OPEN CONJECTURE: } V(K) = 1 \iff K = 0}$$

IDEA: MUTATION:

$K =$

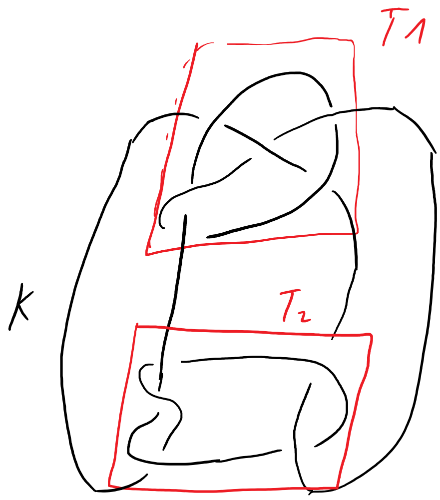


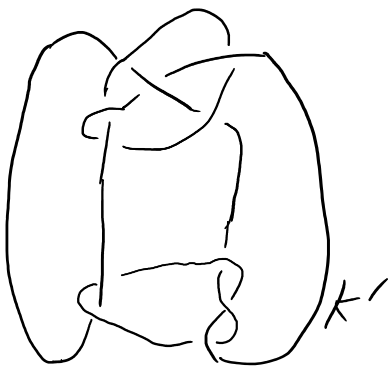
„MUTANT KNOTS ARE NATURAL ENEMIES OF KNOT INVARIANT"

Ex: $V(K) = V(\text{mutant of } K)$

but i.g. $K \not\sim$ mutant of K

Ex:





compute $V(K) = V(K')$

why $K \not\sim K'$?

(at the moment NOT possible!)

Bonus exercise.

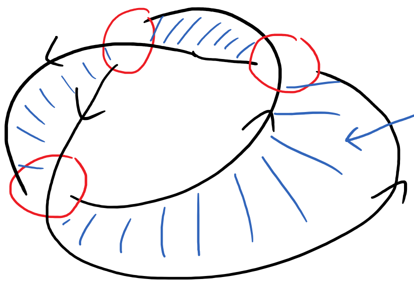
A **Seifert surface** of an oriented link L is an oriented surface embedded surface F in $\mathbb{R}^3$ which intersects the link exactly as its oriented boundary.

(a) Describe an algorithm to produce a Seifert surface of an oriented link from one of its diagrams. *Hint:* First resolve the crossings appropriately and fill the remaining circles by disks. Then try to glue the disks by drilled bands to obtain a Seifert surface of the original link.

(b) The **genus** $g(L)$ of an oriented link is defined to be the minimal genus among all its Seifert surfaces. How does the genus depend on the orientation of the link? Compute the genus for the trefoil and the figure eight knot.

(c) Let K_1 and K_2 be oriented knots. Then $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$. *Remark:* In fact, equality is true. But this is harder to show.

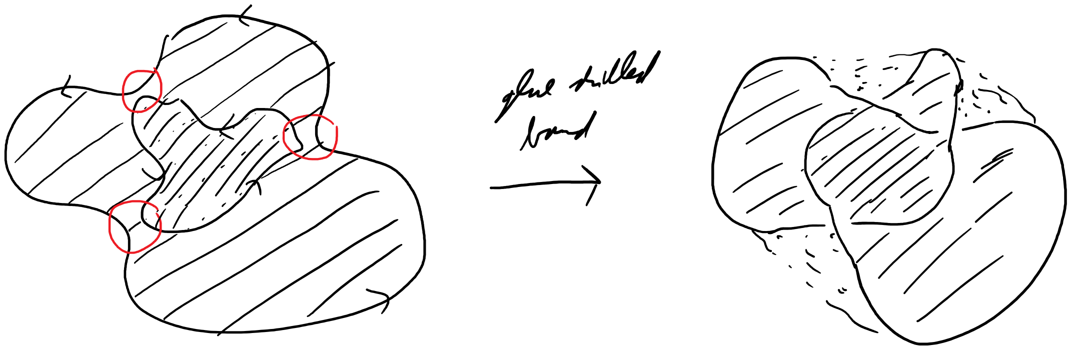
(d) CLAIM: Any oriented knot K admits a SEIFERT SURFACE

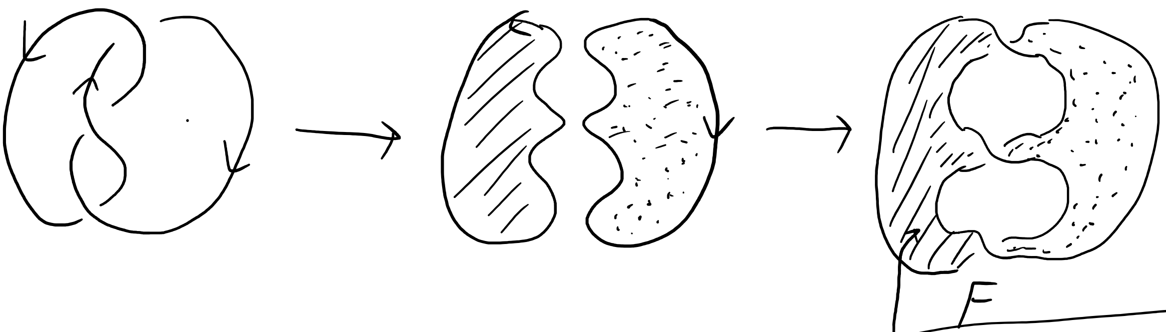


NOT A SEIFERT SURFACE

it is NOT orientable

↓ resolve crossings & glue in disks





$g(F) = ?$

$\chi(F) = 2 \to = -1$

$\Rightarrow F \overset{C^0}{\cong} T^2 \setminus \mathring{D}^2$

$\Rightarrow g(F) = 1$

Seifert surfaces:
https://mathcurve.com/surfaces.gb/seifert/seifert.shtml
https://www.win.tue.nl/~vanwijk/seifertview/tutorial7.htm
https://www.youtube.com/watch?v=px3Gq_gvvac

(b) THE <u>GENUS</u> of an oriented link L

$$g(L) := \min\{\, g(F) \mid F \text{ is a Seifert surface of } L \,\}$$

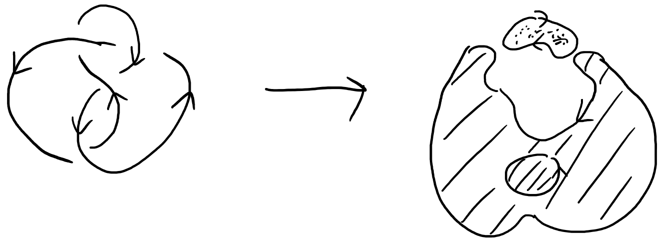
* $g(K) = 0 \iff K = \partial D^2 \iff K = 0$

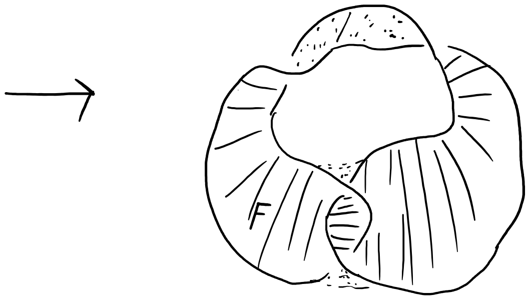
* $g(\text{trefoil}) = 1$

$\ulcorner$ trefoil $\neq 0 \implies g(\text{trefoil}) \geq 1$

$g(F) = 1 \implies g(\text{trefoil}) \leq 1$ $\lrcorner$

* $g(\text{fig8}) = 1$





$\chi(F) = 3 - 4 = -1 \implies F \cong T^2 \setminus \mathring{D}^2$

$\implies g(\text{fig 8}) = 1$

Challenge.

A Brunnian n-link is a non-trivial n-component link consisting of n-unknots, such that removing any of its components yields a trivial $(n-1)$-component link.

(a) Construct for every $n \in \mathbb{N}$ a Brunnian n-link.

(b) Construct infinitely many different 3-component Brunnian links.

Brunnian links:
https://mathcurve.com/courbes3d.gb/brunnien/brunnien.shtml
https://en.wikipedia.org/wiki/Brunnian_link
http://katlas.org/wiki/Brunnian_link

Sheet 3

Exercise 1.

(a) Describe an explicit Morse function of $\mathbb{R}P^2$ inducing a handle decomposition of $\mathbb{R}P^2$ with exactly one 0-handle, one 1-handle and one 2-handle.

(b) Sketch an embedding of the surface Σ_2 of genus 2 into $\mathbb{R}^3$, such that the height function is a Morse function on Σ_2 inducing a handle decomposition of Σ_2 with exactly one 0-handle and exactly one 2-handle.

(c) Draw sketches of all handle cancellations and handle slides in dimensions 1, 2 and 3. Indicate in your sketches also the attaching spheres, the belt spheres, the cores, the cocores and the attaching regions.

(a) Construct: $h: \mathbb{R}P^2 \longrightarrow \mathbb{R}$ Morse, i.e.

$\forall p \in \mathbb{R}P^2$: with $\partial_p h = 0 \Rightarrow \det(H_p h) \neq 0$

$$\begin{tikzcd} M^n \supset U \ni p \arrow[rr, "f"] \arrow[d, "\varphi", "\cong"'] & & \mathbb{R} \arrow[d, "\mathrm{id}", "\cong"'] \\ \mathbb{R}^n \supset V \arrow[rr, "\tilde f := \mathrm{id} \circ f \circ \varphi^{-1}"] & & \mathbb{R} \end{tikzcd}$$

$\partial_p f := \partial_{\varphi(p)} \tilde f$ & $H_p f := H_{\varphi(p)} \tilde f$

Remark: $\partial_p f$, $H_p f$ depend on φ

But $\partial_p f = 0$ & $\det(H_p f) \neq 0$ are independent of φ

In part $p \in \mathrm{Crit}(f) \Leftrightarrow \frac{\partial \tilde f}{\partial x_i} = 0 \quad \forall i$

$\tilde h: \mathbb{R}P^2 = S^2/_{p \sim -p} \longrightarrow \mathbb{R}$, $S^2 \subset \mathbb{R}^3_{(x_1, x_2, x_3)}$

ANSATZ: $h: S^2 \longrightarrow \mathbb{R}$

$$(x_1, x_2, x_3) \longmapsto \sum_{i,j,k} a_{i,j,k} \, x_1^i x_2^j x_3^k$$

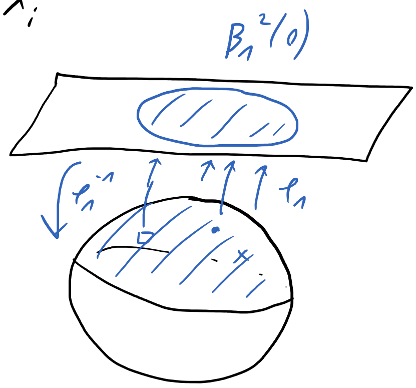
$$\begin{tikzcd} S^2 \arrow[r, "h"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \\ \mathbb{R}P^2 \arrow[ur, dashed, "\tilde h"'] & \end{tikzcd}$$

h induces $\tilde h \Leftrightarrow a_{i,j,k} = 0$ for $i+j+k$ odd

Ex: $h(x) = x_1^2 + 2x_2^2 + 3x_3^2$ $\qquad h(x) = x_3$

ANSATZ: $\quad h : \mathbb{RP}^2 \longrightarrow \mathbb{R}$

$$(x_1, x_2, x_3) \longmapsto \sum_{i=1}^{3} a_i x_i^2$$



Atlas $(U_i, \varphi_i) \quad i = 1, \dots, 3$

$$U_i = \{x_i \neq 0\}$$

$$\varphi_1 : (x_1, x_2, x_3) \longmapsto (u_1, u_2) := \frac{x_1}{|x_1|}(x_2, x_3)$$

$$\varphi_2 : \quad '' \qquad \frac{x_2}{|x_2|}(x_1, x_3)$$

$$\varphi_3 : \quad '' \qquad \frac{x_3}{|x_3|}(x_1, x_2)$$

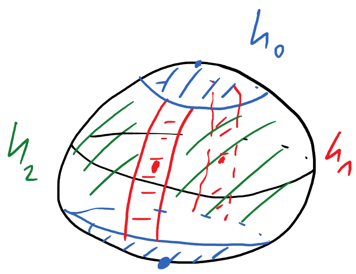
$$\varphi_1^{-1} : (u_1, u_2) \longmapsto \left(\sqrt{1 - u_1^2 - u_2^2}, u_1, u_2\right)$$

$\varphi_2^{-1}, \varphi_3^{-1} \quad \dots$

Compute: $\quad h \circ \varphi_i^{-1} : (u_1, u_2) \longmapsto \sum_{j=1}^{3} a_j x_j^2$

$$= a_i x_i^2 + \sum_{j \neq i} a_j x_j^2$$

$$= a_i (1 - u_1^2 - u_2^2) + \sum_{j \neq i} a_j u_j^2$$



$$= a_i + \sum_{j \neq i} (a_j - a_i) u_j^2$$

$\Rightarrow$ for a_i pairwise different: h is Morse with 3 crit pts at

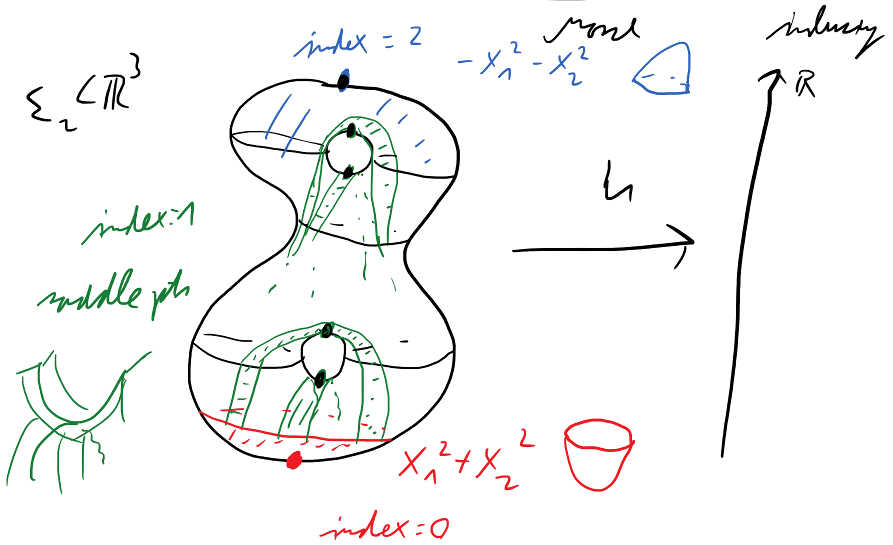
$$\{\varphi_i(0,0)\} = \{[1:0:0], [0:1:0], [0:0:1]\}$$

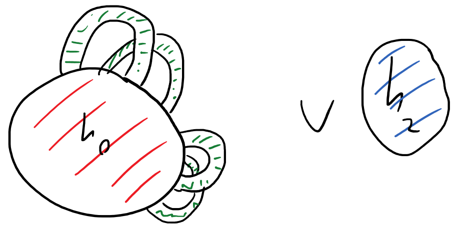
index $0, 1, 2$ (which one has which index depends on order of a_i's)

Index of $p \in \mathrm{Crit}(h)$: $\quad \exists$ local coord x_i: $\quad h(x_1, \dots, x_n) = x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$

$k :=$ index

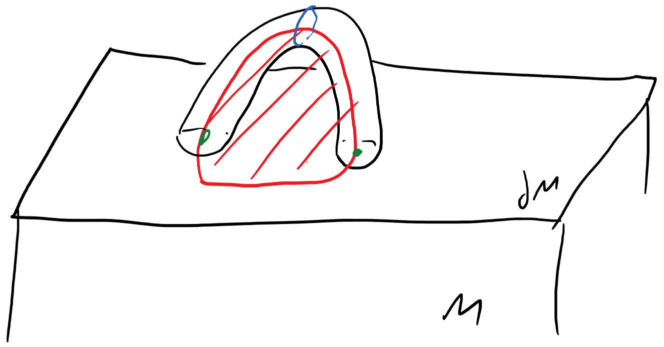
$$\begin{tikzcd} \Sigma_2 \arrow[r, hook] \arrow[rr, bend right] & \mathbb{R}^3 \arrow[r, "\text{height fct}"] & \mathbb{R} \end{tikzcd}$$

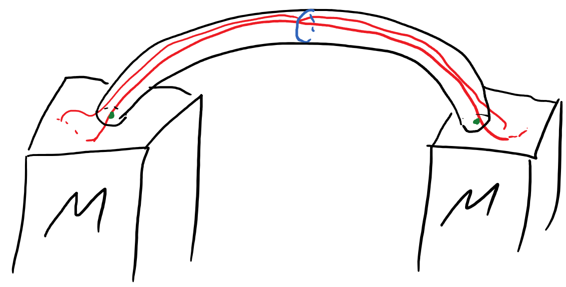




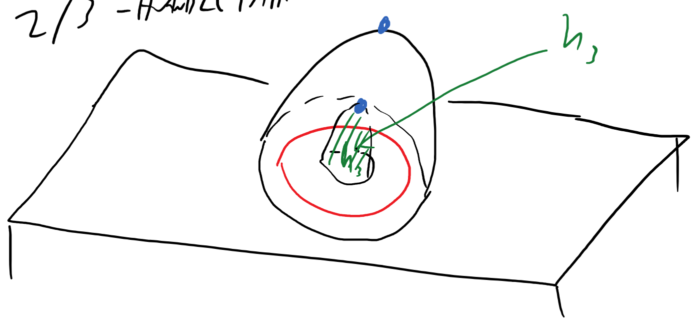
$\mathrm{Crit} = \{ p \in \Sigma_2 \mid T_p\Sigma = (x,y)\text{-plane}$

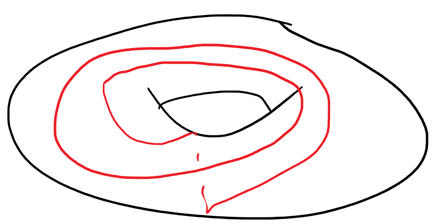
CAN. 1-/2- HANDLE PAIR





2/3 - HANDLE PAIR





Exercise 2.

(a) Use the Alexander trick to show that any manifold obtained by gluing two n-disks is homeomorphic to the n-sphere.

(b) In the lecture we constructed Milnor's exotic 7-sphere E^7 by gluing to copies of $S^3 \times D^4$ via a diffeomorphism of their boundaries. Verify that this construction defines a natural smooth structure on E^7.

(c) Describe an explicit Morse function on E^7 with exactly two critical points and conclude that E^7 is homeomorphic to S^7.
Hint: Consider the suitable scaled real part of the S^3-factor in the first copy of $S^3 \times D^4$ and try to extend that map over the second copy of $S^3 \times D^4$ (where we see S^3 again as in the lecture as the unit sphere in the quaternions). Of course one could also look into Milnor's original paper and just copy the formula and compute that it is a Morse function with the desired properties, but then you will not learn much from this exercise.

(a) $M^n = D^n \cup_\varphi D^n$, $\quad S^n = D^n \cup_{id} D^n$

$$\begin{tikzcd}
\varphi(p) \arrow[d, "\overset{C^0}{\cong}"'] & p \in \partial D^n \arrow[l, mapsto] \arrow[d, "\cong\, id"] \\
p & p \in \partial D^n \arrow[l, mapsto]
\end{tikzcd}$$

$C^0 \cong$ Alex.-trick $\Rightarrow$ extension of φ^{-1}

(b) $E^7 := D^4 \times S^3 \cup_\varphi D^4 \times S^3$

$S^3 \times S^3 \ni (z,w) \xmapsto{\tilde\varphi} (z, z^2 w z^{-1}) \in S^3 \times S^3$

$= (\mathbb{R}^4 \times S^3)_1 + (\mathbb{R}^4 \times S^3)_2 \,/\sim$

$\mathbb{R}^4 \setminus \{0\} \times S^3 \ni p \sim \varphi(p) \in \mathbb{R}^4 \setminus \{0\} \times S^3$

$$p = (z,w) \xmapsto{\varphi} \left(\frac{z}{|z|^2}, \frac{z^2 w z^{-1}}{|z|} \right) = (z', w')$$

(c) <u>ANSATZ</u> $\quad h: E \to \mathbb{R}$

$(\mathbb{R}^4 \times S^3)_1 \ni (z,w) \longmapsto a\,\mathrm{Re}(w)$

$(\mathbb{R}^4 \times S^3)_2 \ni (z', w') \longmapsto$ extension we get:

$$(z,w) \longmapsto \frac{Re(w)}{\sqrt{1+|z|^2}} \quad \text{for } (z,w) \in (\mathbb{R}^4 \times S^3)_1$$

$$(z',w') \longmapsto \frac{Re(z' w'^{-1})}{\sqrt{1+|z' w'^{-1}|^2}} \quad \text{for } (z',w') \in (\mathbb{R}^4 \times S^3)_2$$

well-def, i.e. agree on $\mathbb{R}^4 \setminus \{0\} \times S^3$

$Crit(h) = \{ (z,w) = (0, \pm 1) \}$ & non-deg

$\Rightarrow \; E^7 \overset{C^\infty}{=} h_0 \cup h_7 \overset{C^\infty}{=} D^7 \cup D^7 \overset{C^0}{\cong} S^7$

but $E^7 \overset{C^\infty}{\not\cong} S^7$

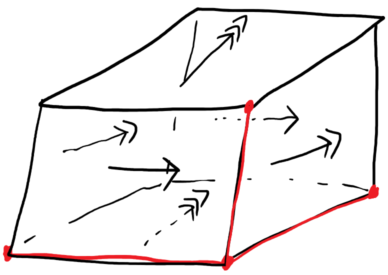
Exercise 3.
We consider the 3-torus $T^3 := S^1 \times S^1 \times S^1$.

(a) Show that we can obtain T^3 from the cube $I \times I \times I$ by identifying opposite sides.

(b) Describe a handle decomposition of T^3 (as simple as possible).

(c) Draw a planar Heegaard diagram of T^3.

(a)



$S^1 \times S^1 \times S^1 = T^3$

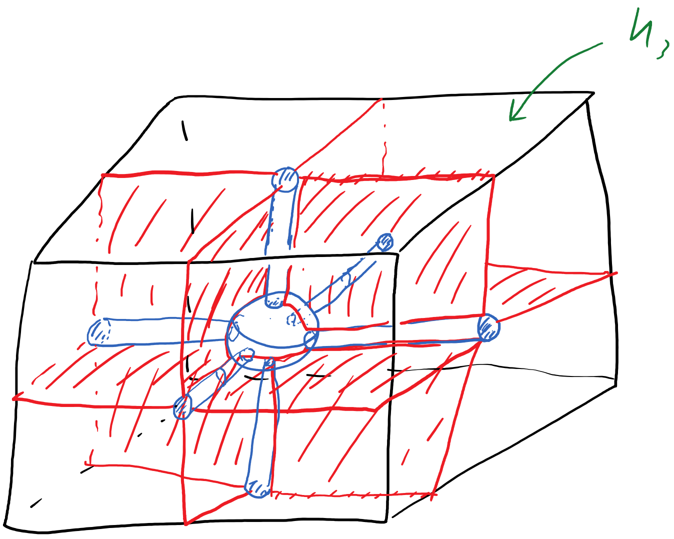
(b) ① Observation: $h_k^{(n)} \times h_\ell^{(m)} = (\underline{D^k} \times D^{n-k}) \times (\underline{D^\ell} \times D^{m-\ell})$

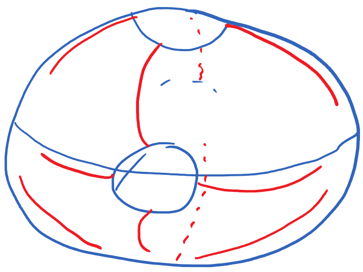
$$= (\underline{D^k \times D^\ell}) \times (D^{n-k} \times D^{m-\ell})$$

$$= \underline{D^{k+\ell}} \times D^{n+m-(k+\ell)}$$

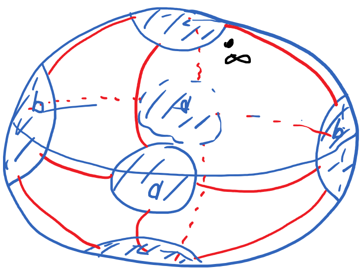
$$= h_{k+\ell}^{(n+m)}$$

(2)





$$T^3 = \underbrace{h_0 \cup h_1^1 \cup h_1^2 \cup h_1^3}_{H_1} \cup \underbrace{h_2^1 \cup h_2^2 \cup h_2^3 \cup h_3}_{H_2}$$



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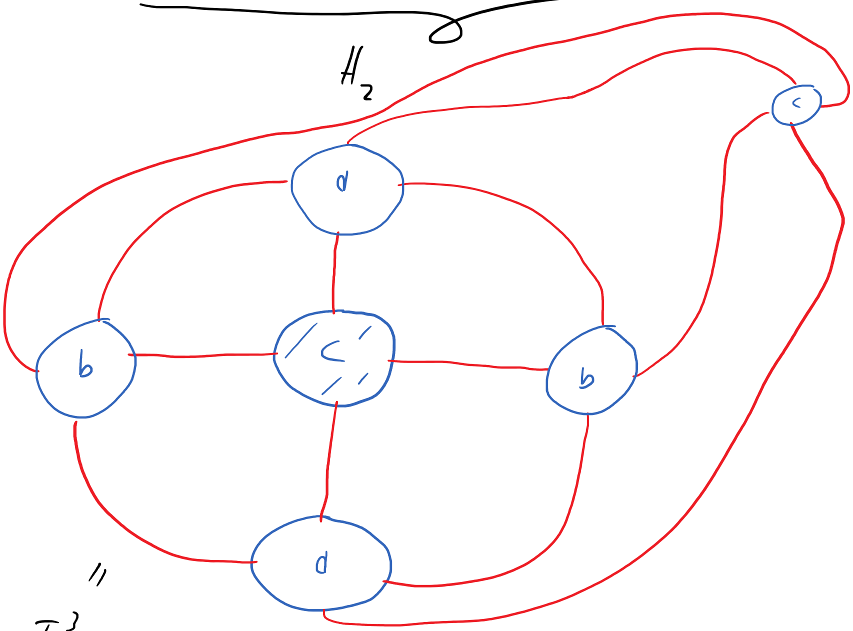


$\shortparallel$

T^3

SIMILAR FOR $S^1 \times \Sigma_g$

$$\pi_1(T^3) = \pi_1(\mathbb{R}^3/\mathbb{Z}^3) = \mathbb{Z}^3$$

$$\neq$$

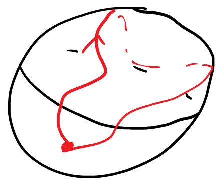
$$\pi_1(L(p,q)) = \pi_1(S^3/\mathbb{Z}_p) = \mathbb{Z}_p$$

$\boxed{\text{Ex } 4}$ $\pi_1(M) = \pi_1(M_2)$ for M connected

$\ulcorner$ Let $k \geq 3 \Rightarrow h_k = D^k \times D^{n-k}$ $\urcorner$

attached along $\partial D^k \times D^{n-k}$

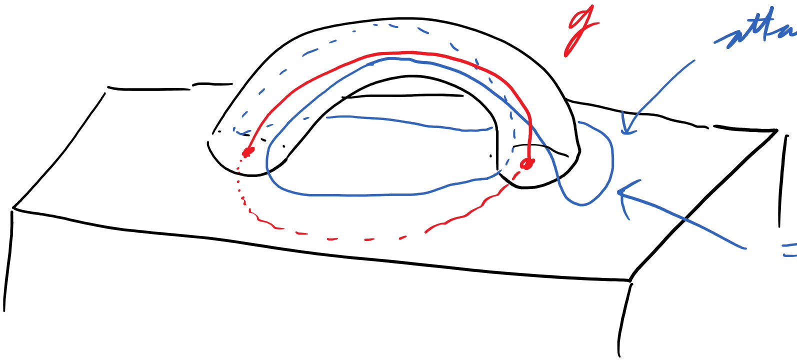
$$\pi_1(\underbrace{\partial D^k \times D^{n-k}}_{S^{2} \text{ or } S^{\ell \geq 2}}) = 1 = \pi_1(h_k)$$



$\overset{\text{SvK}}{\Rightarrow}$ $\pi_1(M \cup h_k) = \pi_1(M)$ $\lrcorner$

$\llcorner$

$$\pi_1(M_2) = \langle h_1^i \mid h_2^j \rangle$$



$$\mathbb{Z}^4 = \pi_1(T^4)$$

$\exists$ 3-mfd M closed s.t. $\pi_1(M) = \mathbb{Z}^4$

Exercise 4.

(a) Describe a way to compute the fundamental group of a manifold with a given handle decomposition.

(b) The fundamental group of a compact smooth manifold is finitely presented. Conversely, we can get for any $n \geq 5$ any finitely presented group as the fundamental group of a closed oriented n-manifold.

Challenge: Can you show the same for $n = 4$?

(c) On the other hand, not every finitely presented group occurs as the fundamental group of a closed orientable 3-manifold. Groups arising as the fundamental group of a closed orientable 3-manifolds are called 3-**manifold groups**.
Hint: Let $\langle g_1, \dots g_n | r_1, \dots r_k \rangle$ be a finite presentation of a group G. We call $n - k$ the deficiency of this presentation. The **deficiency** of a finitely presented group G is the maximum deficiency of a finite presentation for G. Then you need to show that any 3-manifold group has non-negative deficiency and find a group with negative deficiency.

(b) $G = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ fin. pres. of G

we construct M as follows:

(1) start with a 0-handle $h_0 = D^n$

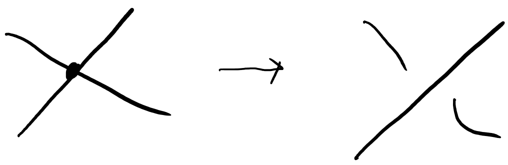
(2) we attach k 1-handles $(n \geq 3)$

$\pi_1\Big($  $\Big) \overset{L1}{=} \pi_1\big(\natural_k S^1 \times D^{n-1}\big) = \langle g_1, \dots, g_k \rangle$

(3) * write r_j as words in g_i

* Realize the words r_j as disjoint simple closed curves in $\partial(\natural_k S^1 \times D^{n-1})$

$n \geq 4 \Rightarrow \dim(\partial \natural_k S^1 \times D^{n-1}) \geq 3$



TRANSV. THM:

$U, V \subset M$

$\Rightarrow \exists$ pert $\tilde{U}, \tilde{V}$ s.t. $\tilde{U} \pitchfork \tilde{V}$

i.e. $\forall p \in \tilde{U} \cap \tilde{V}$: $T_p\tilde{U} + T_p\tilde{V} = T_pM$

$\dim(U) = 1 = \dim(V)$

$\dim(M) \geq 3$

* Attach 2-handles along the curves representing r_j

$\Rightarrow \pi_1(M_2) = G$

(4) $M := DM_2 := M_2 \cup M_2$

DM_2 has a handle decomp:

$n=4 \to 2$-handle

$$\underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^k \cup h_2^1 \cup \dots \cup h_2^l}_{M_2} \cup \underbrace{h_{n-2}^1 \cup \dots \cup h_{n-2}^l \cup h_{n-1}^1 \cup \dots \cup h_{n-1}^k \cup h_n}_{\text{dual handl. dec. of } M_2}$$

$n \geq 5 \Rightarrow h_{n-2}$ index ≥ 3

$\Rightarrow \pi_1(M) = \pi_1(M_2) = G$

$$h : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(x, y, z) \longmapsto \sin(x) + \sin(y) + \sin(z)$$

$$\longrightarrow h : T^3 \longrightarrow \mathbb{R}$$

$$J_p h = \begin{pmatrix} \cos(x) \\ \cos(y) \\ \cos(z) \end{pmatrix} = 0$$

$$\Leftrightarrow \quad x, y, z \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \qquad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3}{2}\pi \right)$$

$$H_p h = \begin{pmatrix} -\sin(x) & 0 & 0 \\ 0 & -\sin(y) & 0 \\ 0 & 0 & -\sin(z) \end{pmatrix}$$

$$\det = -\sin(x)\sin(y)\sin(z) \neq 0$$

Sheet 4

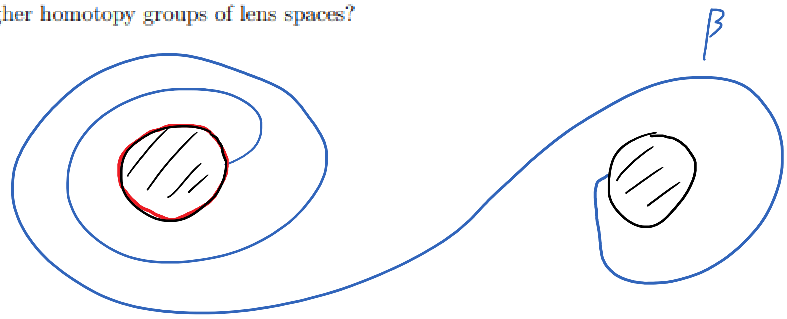
Exercise 1.
Let M be a connected closed orientable 3-manifold presented by a Heegaard diagram.

(a) Conduct a presentation of the first homology group $H_1(M;\mathbb{Z})$ only depending on the homological information of the Heegaard diagram.

(b) Describe a presentation of the fundamental group of M.

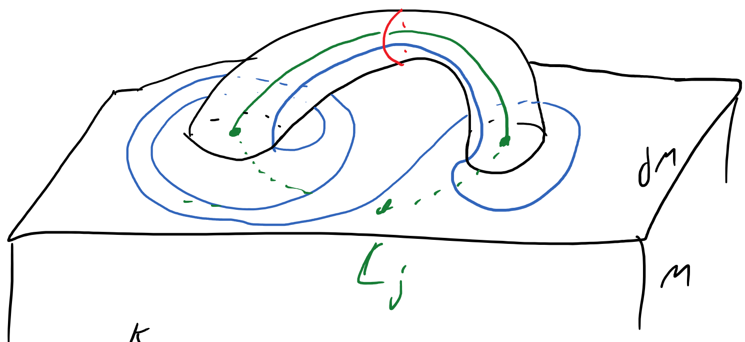
(c) Compute the fundamental group and homology groups of the lens spaces $L(p,q)$ from their Heegaard diagrams. What are the higher homotopy groups of lens spaces?

(a) Heeg. diag



$\beta_i :=$ attaching sphere of h_2^i

$B_j :=$ belt sphere of h_1^j

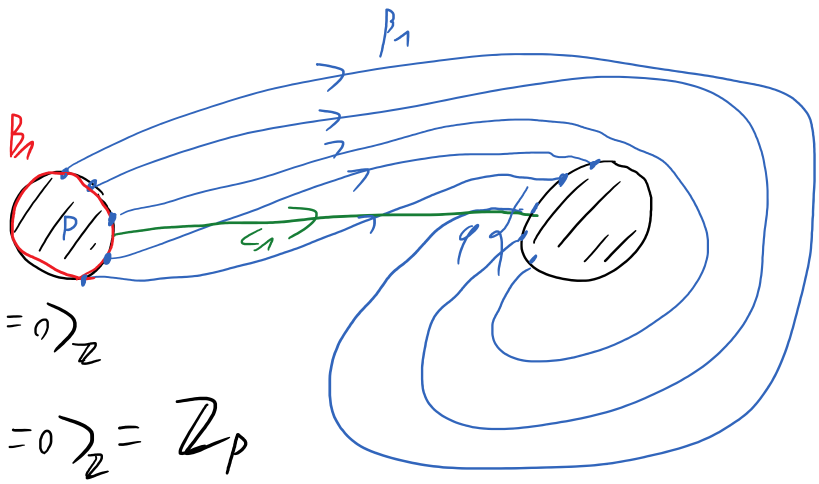


$$H_1(M_1) = \langle c_j \rangle_{\mathbb{Z}} \cong \mathbb{Z}^k \qquad k = \#\text{ 1-handles}$$

$$\beta_j = \sum_{i=1}^{k} \underbrace{(B_i \cdot \beta_j)}_{\text{intersection product of } B_i, \beta_j \text{ in } \partial M_1} c_i$$

$$\boxed{H_1(M) = \langle \underset{\substack{\equiv \\ c_i}}{h_1^i} \mid \sum_{i=1}^{k} (B_i \cdot \beta_j)\, h_1^i \rangle}$$

(c) Ex: $L(p,q) \overset{T.8}{=}$



$$H_1(L(p,q)) = \langle h_1 \mid (B_1 \cdot \beta_1) h_1 = 0 \rangle_{\mathbb{Z}}$$
$$= \langle h_1 \mid p\, h_1 = 0 \rangle_{\mathbb{Z}} = \mathbb{Z}_p$$

Rem: $H_0(M^3) = H_3(M^3) = \mathbb{Z}$ con., or., closed

$H_2(M) = H^1(M) = F_1$

$H_2(L(p,q)) = 0$ for $p \neq 0$

* $L(p,q) = S^3/\mathbb{Z}_p \Rightarrow \pi_1(L(p,q)) = \mathbb{Z}_p$

$\pi_k(L(p,q)) = \pi_k(S^3) \quad \forall k \geq 3$

(b) $\pi_1(M) = \langle c_i \;/\; \beta_j \rangle$

$\uparrow$ seen as a curve in M_1

Ex: $\pi_1(L(p,q)) = \langle c_1 \;/\; c_1^p \rangle \cong \mathbb{Z}_p$

Exercise 2.
Let M and N be two connected, smooth, oriented, closed n-manifolds. The **connected sum** $M\#N$ is the closed, oriented n-manifold defined as follows. Choose embeddings $i_M: D^n \to M$ and $i_N: D^n \to N$, where i_M preserves the orientation and i_N reverses the orientation. The connected sum is obtained from

$$(M \setminus i_M(0)) + (N \setminus i_N(0))$$

by identifying points $i_M(tp)$ with points $i_N((1-t)p)$ for $p \in S^{n-1}$ and $0 < t < 1$.

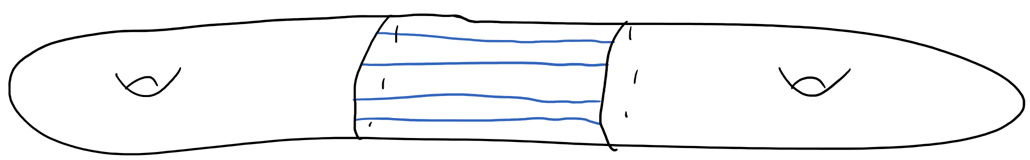
(a) It is possible to show that this is a well-defined operation. (This uses methods from differential topology and is not your task.) What would you have to show for it?

(b) Let M and N be two connected, smooth, compact, oriented n-manifolds with non-empty connected boundary. The **boundary connected sum** $M\natural N$ is obtained from M and N by attaching an 1-handle to the boundary of M and N such that the resulting manifold is oriented and connected. Show that this is well-defined and that we have $\partial(M\natural N) = \partial M\#\partial N$.

(c) Show that the Heegaard genus is sub-additive under connected sum, i.e. show that

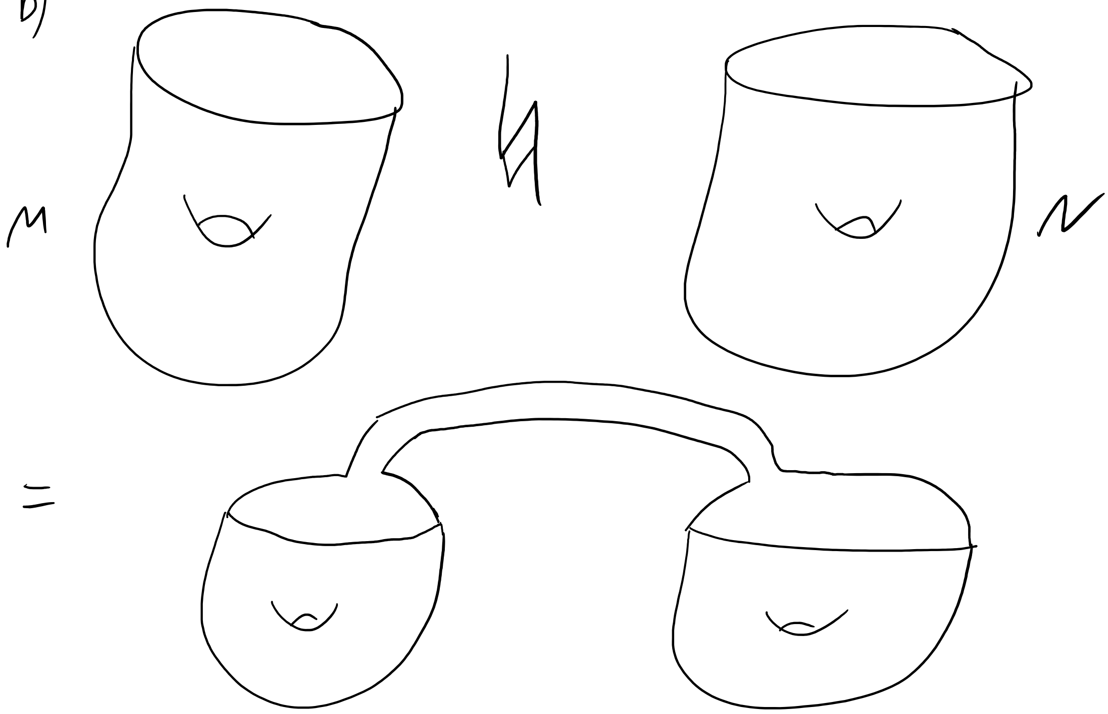
$$g(M\#N) \leq g(M) + g(N)$$

holds. To do this, figure out how to get a Heegaard diagram of $M\#N$ from Heegaard diagrams of M and N.



(d) well-def: by DISK THM: any two disks in a connected mfd are isotopic

(b)

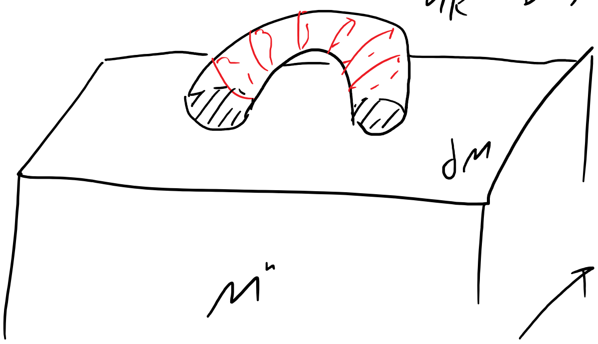


* well def: as in proof of L.3.1.

CLAIM: $\partial(M \natural N) \cong \partial M \# \partial N$

"ATTACHING A HANDLE $\hat{=}$ SURGERY AT THE BOUNDARY"

$h_k = D^k \times D^{n-k}$



In ∂M:

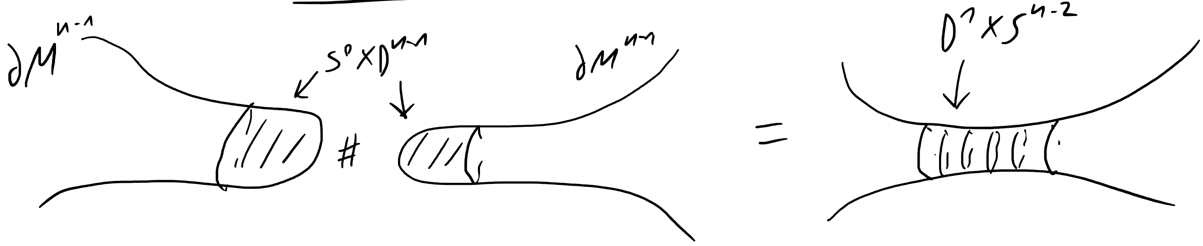
REMOVE: $\partial D^k \times D^{n-k} \cong S^{k-1} \times D^{n-k}$

RE-GLUE: $D^k \times \partial D^{n-k} \cong D^k \times S^{n-k-1}$

SURGERY

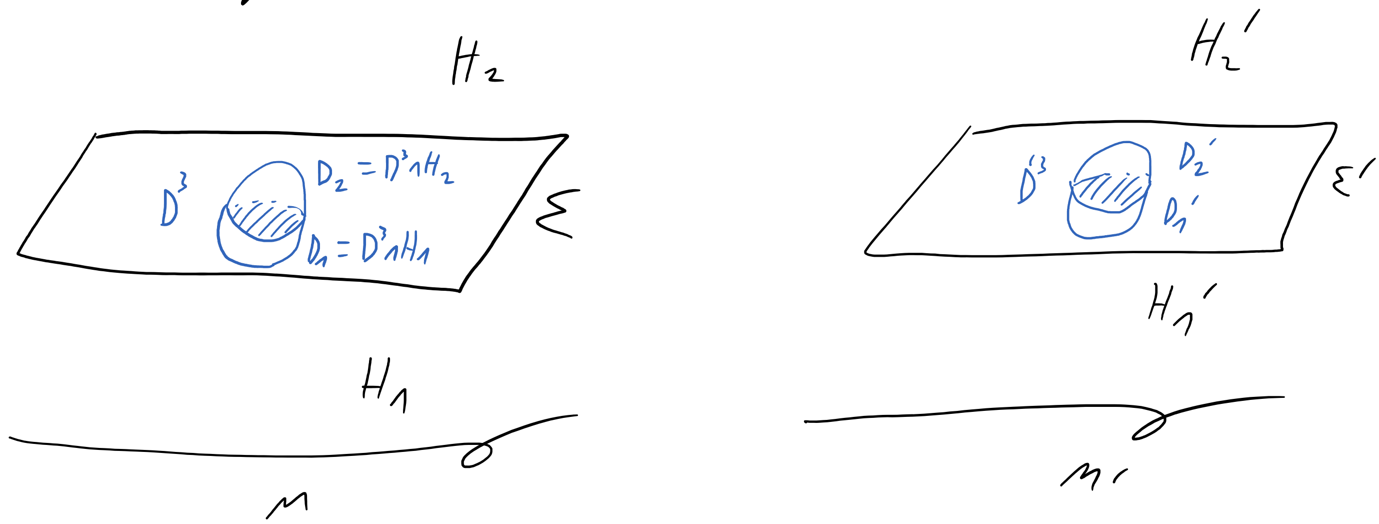
for $k=1$: REMOVE $S^0 \times D^{n-1} = D^{n-1} \sqcup D^{n-1}$

RE-GLUE $D^1 \times S^{n-2}$



(c) Let M, M' be closed, or., con. 3-mfds

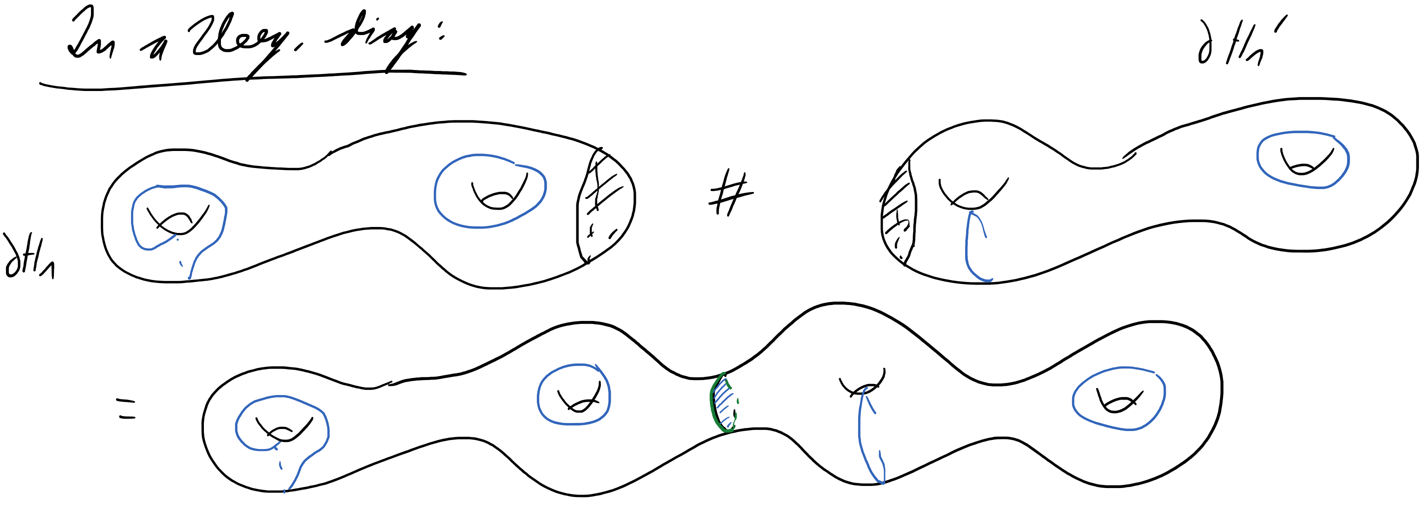
<u>CLAIM</u>: $g(M \# M') \leq g(M) + g(M')$



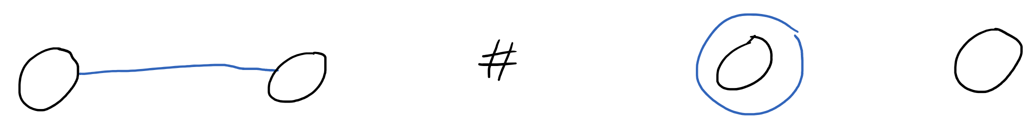
$$M \# M' = (H_1 \cup_\Sigma H_2) \# (H_1' \cup_{\Sigma'} H_2')$$

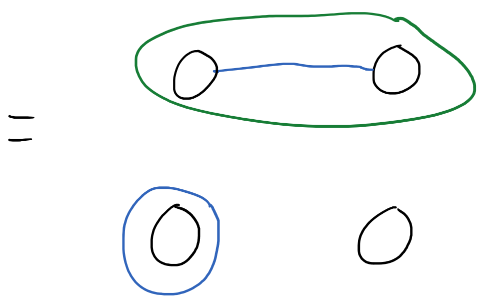
$$(H_1 \setminus D_1 \cup H_1' \setminus D_1') \cup_{\Sigma \# \Sigma'} (H_2 \setminus D_2 \cup H_2' \setminus D_2')$$

<u>In a Heeg. diag:</u>



<u>planar</u>:





<u>Remark</u>: STABILIZATION

$\hat{=}$ # with 

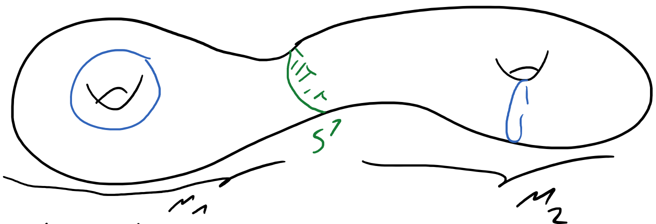
Let (Σ, β_i), (Σ', β_i') be Heeg. diag. of min genus of M & M'. g, g'

$\Rightarrow$ $(\Sigma \# \Sigma', \beta_i \cup \beta_i')$ is a Heeg. diag. of $M \# M'$ of genus $g+g'$

$\Rightarrow$ $g(M \# M') \leq g(M) + g(M')$ $\blacksquare$

Remark: A Heeg. splitting (Σ, β_i) is called REDUCIBLE

$:\Leftrightarrow$ $\exists S^1$ on Σ bounding a disk in H_1 & H_2



$\Rightarrow$ $(\Sigma, \beta_i) = M_1 \# M_2$

HAKEN'S THM Let M be a REDUCIBLE, i.e. $M = M_1 \# M_2$ (with $M_1, M_2 \neq S^3$).

$\Rightarrow$ $\forall$ Heegaard splitting of M is reducible

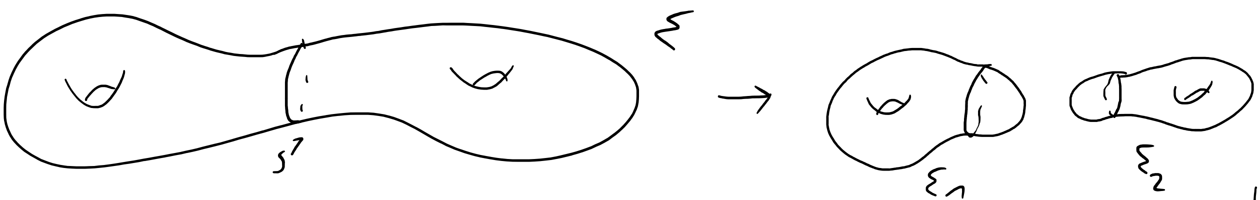
See: https://www2.mathematik.hu-berlin.de/~kegemarc/Kirby/Hausarbeit%20Lennart%20Struth.pdf

Corollary: $g(M_1 \# M_2) = g(M_1) + g(M_2)$

⌜Let Σ be a H.S. of $M_1 \# M_2$ $\overset{H.T.}{\Rightarrow}$ $\exists S^1 \subset \Sigma$ s.t. the HS is reducible ⌝

$\Rightarrow$ Σ_1 & Σ_2 H.S. of M_1 & M_2

s.t. $g(\Sigma_1) + g(\Sigma_2) = g(\Sigma)$



⌞ ⌟

Corollary: $\exists$ PRIM FACTOR DECOMP OF 3-MFDS

$M = M_1 \# \dots \# M_k$ s.t. M_i are PRIME

(i.e. $M_i = N \# N' \Rightarrow N$ or $N' = S^3$)

Remark: the decomp is unique. (MILNOR)

Exercise 3.

(a) The Heegaard genus of T^3 is 3.
Hint: Consider the first homology or the fundamental group of T^3.

(b) A bit more general, construct for any natural number g a 3-manifold with Heegaard genus g

(c) The Heegaard genus of $\Sigma_g \times S^1$ is equal to $2g+1$.

Bonus: The Heegaard genus of a surface bundle of a surface Σ_g of genus g over S^1 is equal to $2g+1$. Where a surface bundle over S^1 is defined as follows. We start with a surface Σ_g of genus g and a diffeomorphism $\phi\colon \Sigma_g \to \Sigma_g$. Then the **surface bundle** over S^1 with **monodromy** ϕ is defined as the quotient space $\Sigma \times I/\sim$ where $(p,1)\sim(\phi(p),0)$.

(a) CLAIM: $g(T^3) = 3$

$g(T^3) \le 3$ [Heeg. diag of T^3 with $g=3$]

$g(T^3) \ge 3$

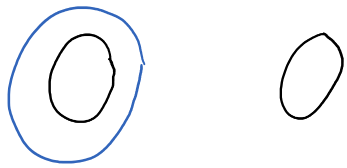
[i.g. $H_1(M = H_1 \cup_{\Sigma_g} H_2) = \langle h_1^1, \dots, h_1^g \mid \beta_1, \dots, \beta_g \rangle_{\mathbb{Z}}$

$\Rightarrow \mathrm{rk}(H_1(M)) \le g$

$\Rightarrow \mathrm{rk}(H_1(M)) \le g(M)$

here: $H_1(T^3) = \mathbb{Z}^3 \Rightarrow 3 \le g(T^3)$]

(b) $g(S^1 \times S^2) = 1$ [



$\Rightarrow g(S^1\times S^2) \le 1$

$\mathrm{rk}(H_1(S^1\times S^2)) = 1 \Rightarrow g(S^1\times S^2) \ge 1$]

$$g(\#_k S^1\times S^2) = k$$

$\#_k S^1\times S^2 =$

$\Rightarrow g(\#_k S^1\times S^2) \leq k$

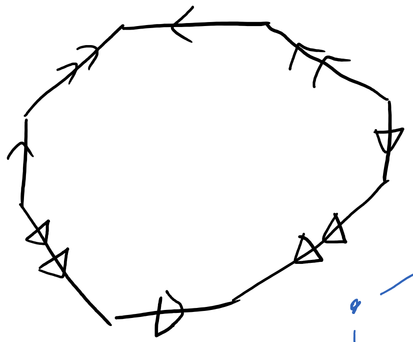
$$H_1(\#_k S^1\times S^2) = \left\langle h_1^1,\dots,h_1^k \;\middle|\; \sum_i \underbrace{(B_i\cdot\beta_j)}_{=0}\, h_1^i = 0 \right\rangle_{\mathbb{Z}} = \mathbb{Z}^k$$

$\Rightarrow \mathrm{rk}(H_1(\#_k S^1\times S^2)) = k \leq g(\#_k S^1\times S^2)$

(c) $H_1(\Sigma_g \times S^1) = \mathbb{Z}^{2g+1}$

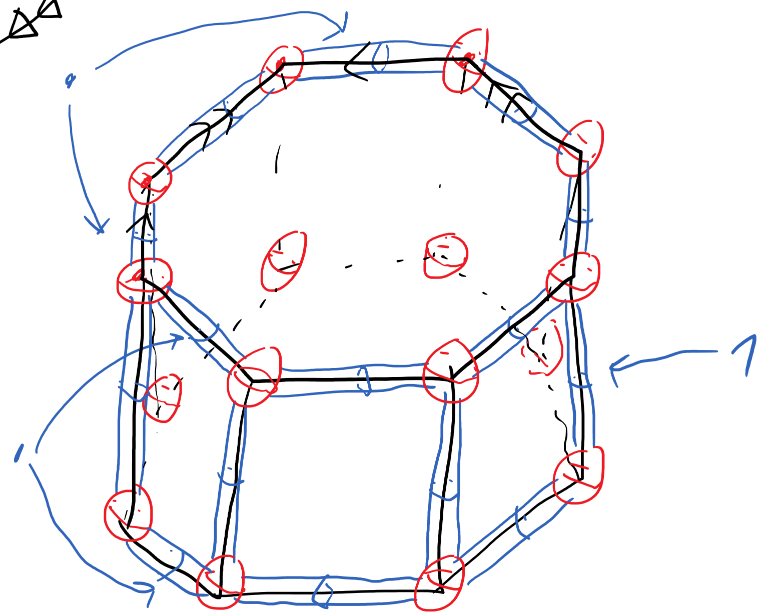
$\Rightarrow rk(H_1(\Sigma_g \times S^1)) = 2g+1 \leq g(\Sigma_g \times S^1)$

Σ_g can be presented by a $4g$-gon with edges identified:



$= \Sigma_2$

$\Sigma_g \times S^1$:

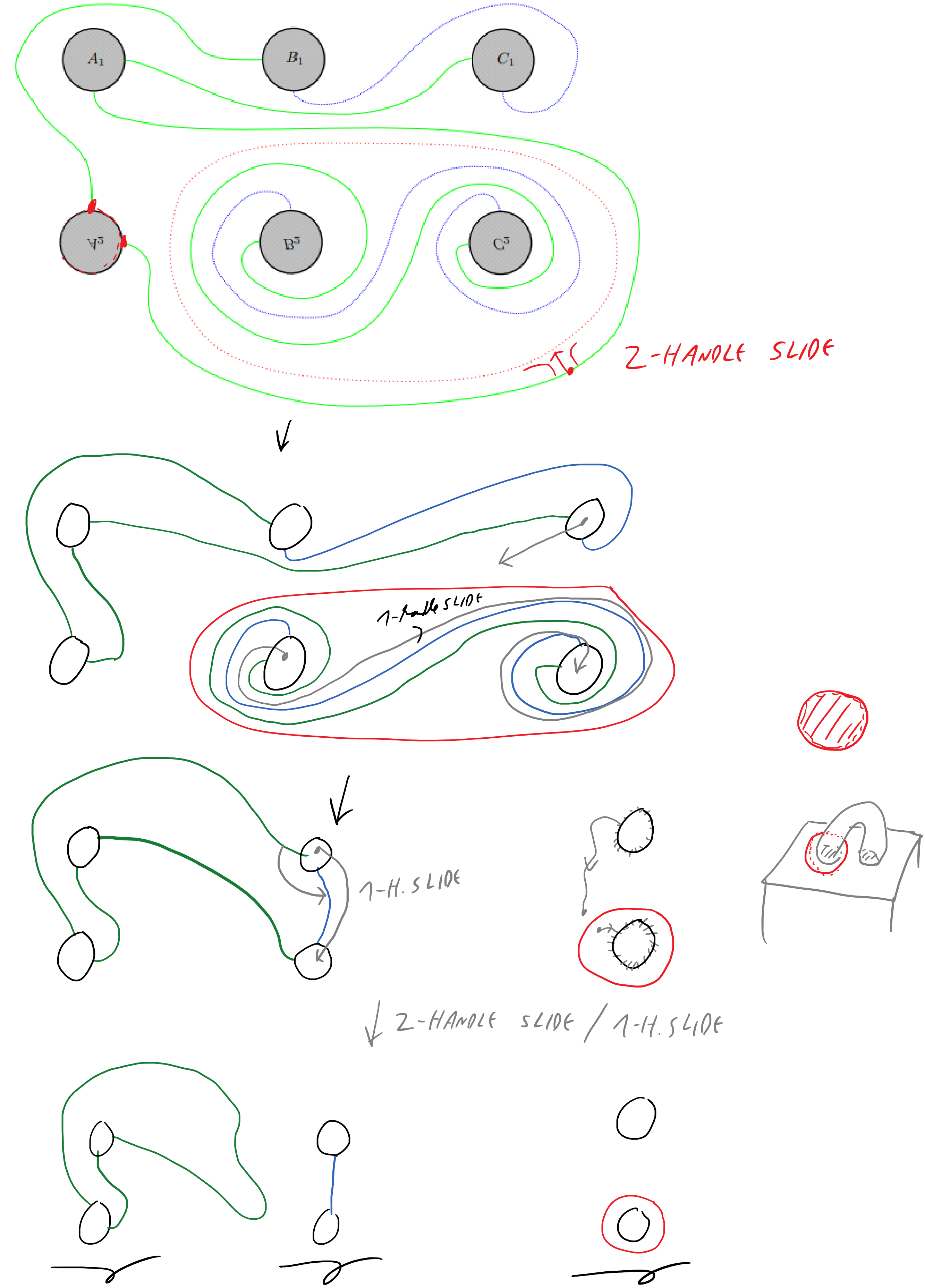


one h_0

$2g+1$ h_1

$\Rightarrow$ Heeg. split of genus $2g+1$

Exercise 4.
Which 3-manifold is presented by the following planar Heegaard diagram?



$\mathbb{RP}^3 = L(2,1) \quad \# \quad S^3 \quad \# \quad S^1 \times S^2 \quad = \mathbb{RP}^3 \# S^1 \times S^2$

SHEET 5

Exercise 1.

Let K be a knot in a connected closed oriented 3-manifold M.

(a) There exists a Heegaard splitting of M such that K lies on its Heegaard surface.

(b) Compute the homology class of K in $H_1(M;\mathbb{Z})$ from a Heegaard splitting $(\Sigma_g;\beta_1,\dots,\beta_g)$ of M swith $K\subset\Sigma_g$.

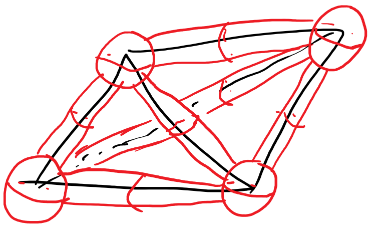
(c) Describe non-nullhomologous knotss in planar Heegaard diagrams of the lens spaces $L(p,1)$ and $S^1\times S^2$. Which homological order have these knots? Show that these knots do **not** admit Seifert surfaces.

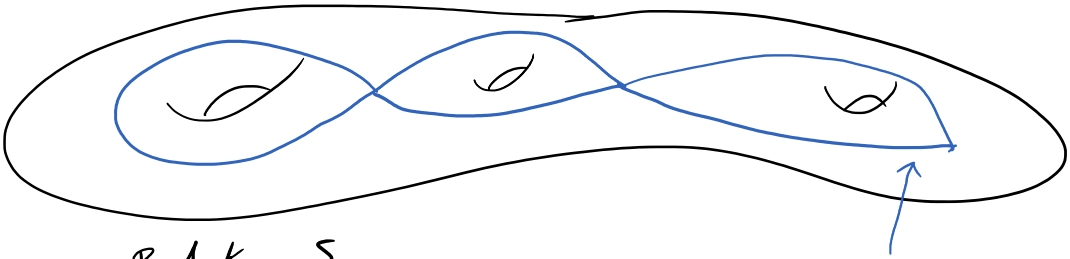
Remark: Later we will show, that a knot admits a Seifert surface if and only if it is nullhomologous.

(a) Choose a triangulation of M s.t. $K\subset$ 1-skeleton

→ get a Heeg.-split. of M s.t.

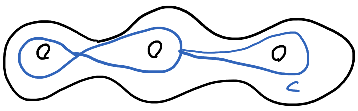
$K\subset$ „core of H_1"





→ Push K on Σ

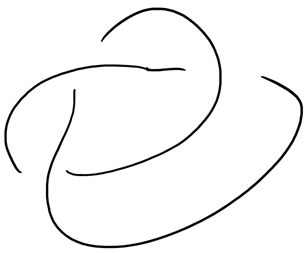
$c\times\{1/2\}\supset K$

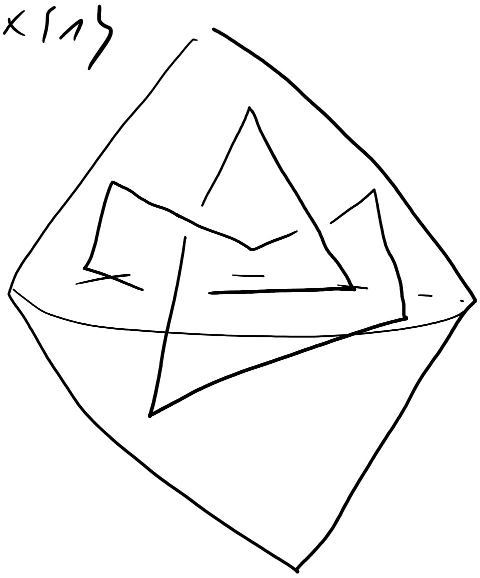
$H_1 =$  $\times I$

$D^2\setminus\bigcup D^2_i$

$K\times\{1\}$

$K \sim$ PL knot

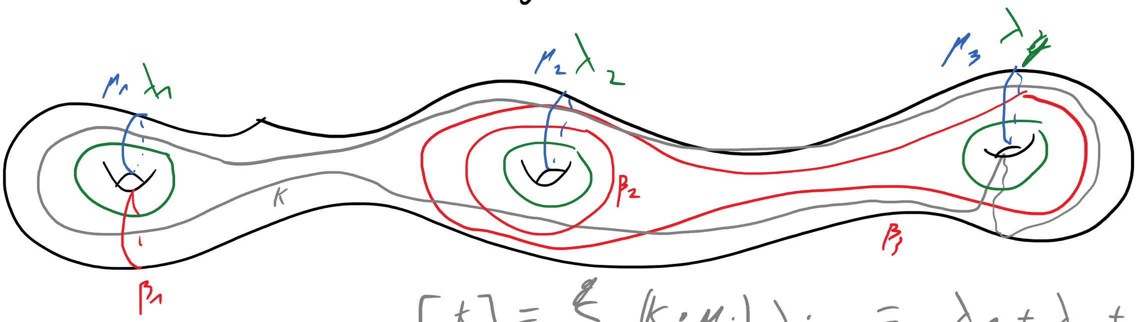




(b) $H_1(M) = \langle \lambda_1, \dots, \lambda_g \mid \beta_1, \beta_2, \beta_3 \rangle$

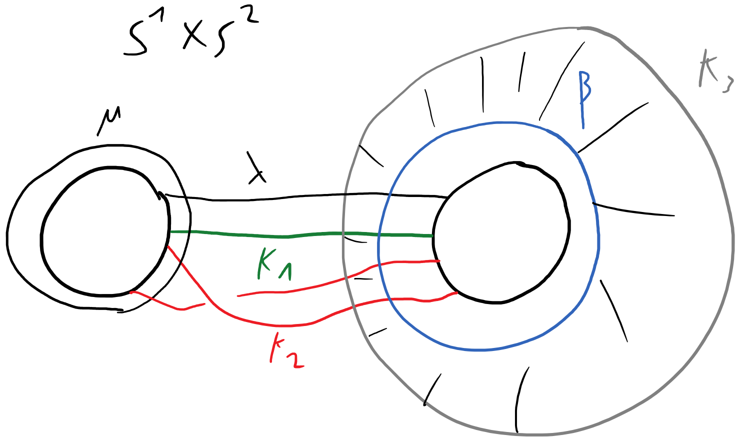
$= \langle \lambda_1, \dots, \lambda_g \mid \sum_{j=1}^{g} (\beta_i \cdot \mu_j)\, \lambda_j \rangle$

$\beta_i \cdot \lambda_j = \delta_{ij}$



$[K] = \sum_{j=1}^{g} (K \cdot \mu_j)\, \lambda_j = \lambda_1 + \lambda_2 + \lambda_3 \in H_1(M)$

(c) $S^1 \times S^2$



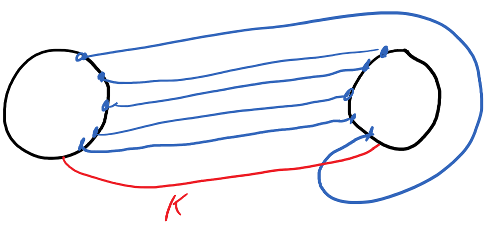
$H_1(S^1 \times S^2) = \langle \lambda \mid \emptyset \rangle \cong \mathbb{Z}_{\langle \lambda \rangle}$

$[K_1] = [\lambda] = \pm 1$

$[K_2] = \pm 2$

$[K_3] = 0$

$L(p,1)$



Alle same

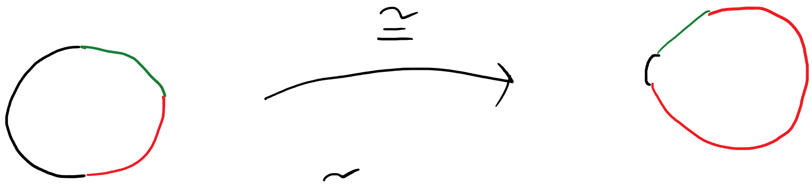
Exercise 2.

(a) Any orientation preserving homeomorphism of S^1 is isotopic to the identity.

(b) Let V be a solid torus. A homeomorphism of ∂V extends to a homeomorphism of V if and only if the meridian μ gets mapped to a curve which is isotopic to $\pm\mu$.

(c) A Dehn twist along a non-separating curve on ∂V is not isotopic to the identity, i.e. represents a non-trivial element in the mapping class group.

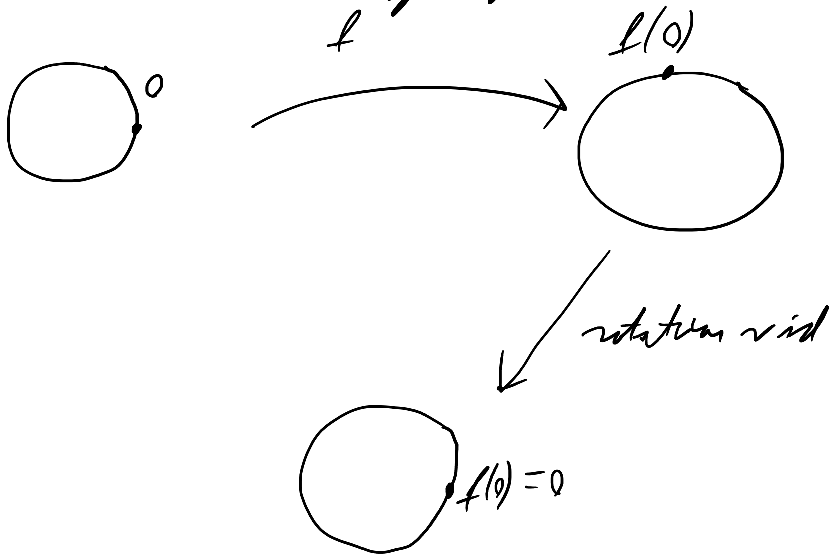
(a)



CLAIM: $f: S^1 \xrightarrow{\cong} S^1$ or preserving $\Rightarrow f \simeq id$

Proof: $S^1 = \mathbb{R}/\mathbb{Z}$ after isotopy by a rotation:

$f(0) = 0$

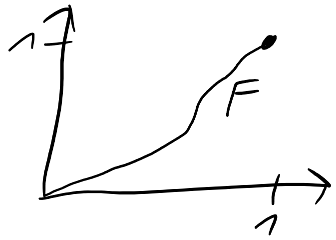


$$\begin{tikzcd} \mathbb{R} = \tilde{S}^1 \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f", "\cong"'] & S^1 \end{tikzcd}$$

Let $F: \mathbb{R} \to \mathbb{R}$ be a lift of f i.e. $\pi \circ F = f \circ \pi$

s.t. $F(0) = 0$

$\Rightarrow F(1)=1$ & $F: I \xrightarrow{\cong} I$



$\Rightarrow$ F strictly increasing

$F_t : x \longmapsto tF(x) + (1-t)x$

is an isotopy from id_I to $F|_I$

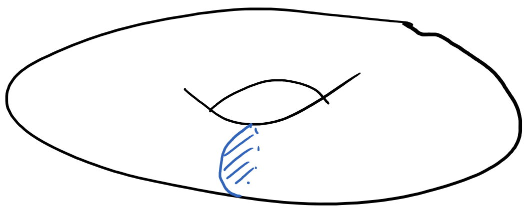
induces an isotopy from id_{S^1} to f. $\blacksquare$

(b) $V \cong S^1 \times D^2$ $\quad f: \partial V \xrightarrow{\cong} \partial V$

CLAIM: f extends to $F: V \xrightarrow{\cong} V \iff f(\mu) \sim \mu$

Proof: μ = non trivial s.c.c. on ∂V s.t. μ is trivial in V

$p\mu + q\lambda$



$\mu \neq 0 \in \partial V$ $\quad \mu = 0$ in V

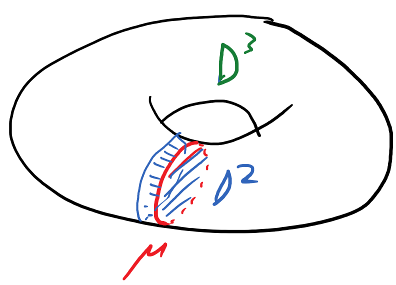
"$\Rightarrow$" f extends to $F: V \xrightarrow{\cong} V$ $\Rightarrow F(\mu) = F(pt \times \partial D^2)$

$\sim pt \times \partial D^2 = \mu$

"$\Leftarrow$" $f(\mu) \sim \mu$

After isotopy $f|_\mu = id$

$\Rightarrow f|_\mu : S^1 \xrightarrow{\cong} S^1$



$\rightarrow$ extend f via Alex. trick over the meridional disk D^2

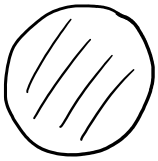
$\rightarrow F : \partial V \cup D^2 \xrightarrow{\cong} \partial V \cup D^2$

$\rightarrow$ extend F via Alexander trick over D^3

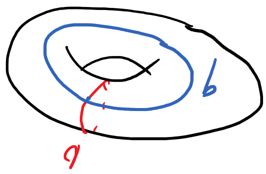
(c) <u>CLAIM</u>: let a be a non-trivial (<u>non-separating</u>) s.c.c. on F^2.

$\Rightarrow T_a \not\simeq id$

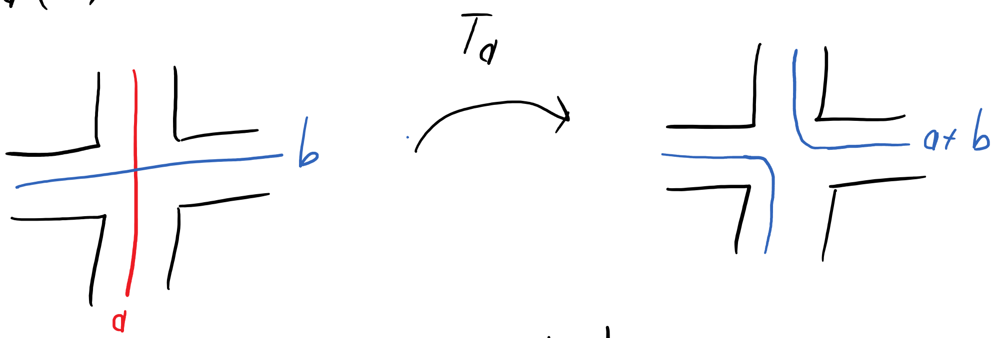
<u>Proof</u>: a non-separating



$\Rightarrow \exists$ s.c.c. b s.t. $a \pitchfork b = \{pt\}$



$T_a(b) = a + b \quad \in H_1(F)$



$b \neq a + b \quad \in H_1(F)$

* $b \cdot b = 0 \quad (F \text{ or.})$

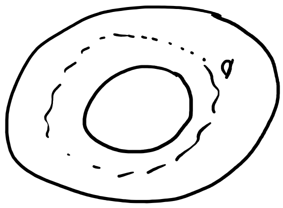
* $(a+b)\cdot b = a \cdot b + b \cdot b = \pm 1$

∎

<u>Rmk</u>: the statement is also true for sep. curves.

Exercise 3.
Determine the isomorphism type of the mapping class group of the annulus $S^1 \times I$ and the 2-torus T^2.

$A = S^1 \times I$



$a = S^1 \times \{1/2\}$

CLAIM: $MCG(A) \cong \mathbb{Z} \langle T_a \rangle$

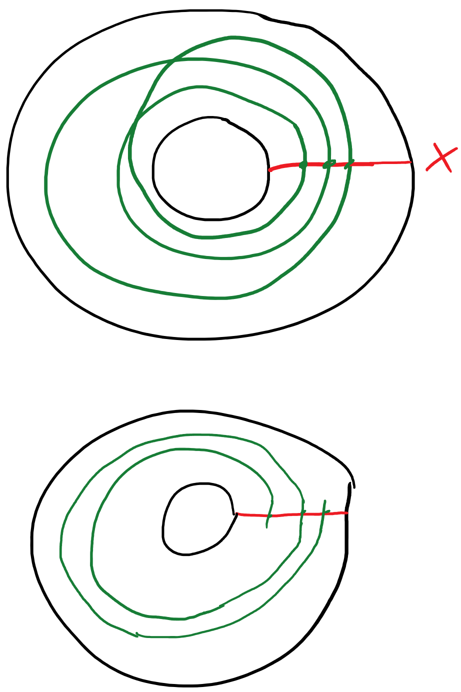
Proof: $MCG(A)$ = gen by Dehn twist along s.c.c.

$\exists !$ non-trivial s.c.c. on A (a)

$\ulcorner$ $H_1(A) \cong \mathbb{Z}$

$c \longmapsto c \cdot x$

* $c = 0 \iff c = \partial D^2 \Rightarrow T_c = id$
* $c = \pm 1 \iff c \sim \pm a \Rightarrow T_a$
* $|c| > 1$ c has self intersections



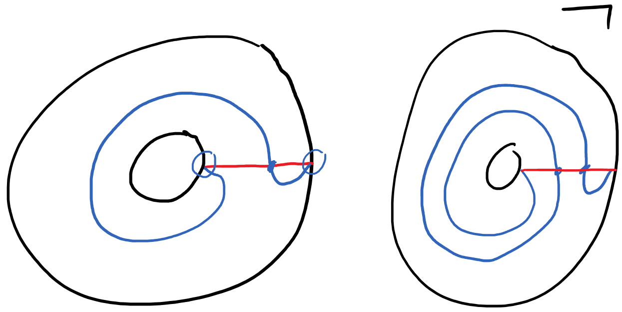
$\llcorner$

* $T_a^{\,n} \neq T_a^{\,m}$ for $n \neq m$:

$\ulcorner$ $T_a^{\,n}(x) = x + na \in H_1(A, \partial A)$

$\neq$

$T_a^{\,m}(x) = x + ma \in H_1(A, \partial A)$



$\llcorner$

CLAIM: $MCG(T^2) \cong SL_2(\mathbb{Z})$

proof:

$$\tau : MCG(T^2) \longrightarrow SL_2(\mathbb{Z})$$

$$[\phi : T^2 \xrightarrow{\cong} T^2] \longmapsto [\phi_* : \underset{\mathbb{Z}^2_{\langle \mu,\lambda\rangle}}{\overset{}{\underset{\wr\!\!\parallel}{H_1(T^2)}}} \xrightarrow{\cong} \underset{\mathbb{Z}^2_{\langle \mu,\lambda\rangle}}{\underset{\wr\!\!\parallel}{H_1(T^2)}}]$$

* well-def: $\phi \in \text{Homeo}^+ \Rightarrow \phi_* \in SL_2(\mathbb{Z})$
  - $\phi \sim \phi' \Rightarrow \phi_* = \phi'_*$

* surjectivity: let $A \in SL_2(\mathbb{Z})$ $\qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix}$$

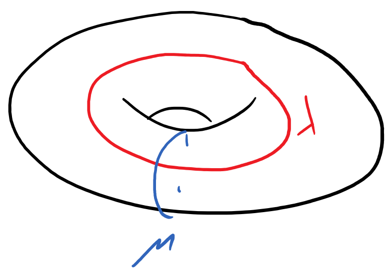
$$\phi_A : T^2 = \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow T^2 = \mathbb{R}^2/\mathbb{Z}^2$$
$$\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] \longmapsto \left[A\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$\Rightarrow \tau : [\phi_A] \longmapsto A$$

* injectivity: let $\phi : T^2 \xrightarrow{\cong} T^2$ s.t. $\phi_* = id_{\mathbb{Z}^2}$

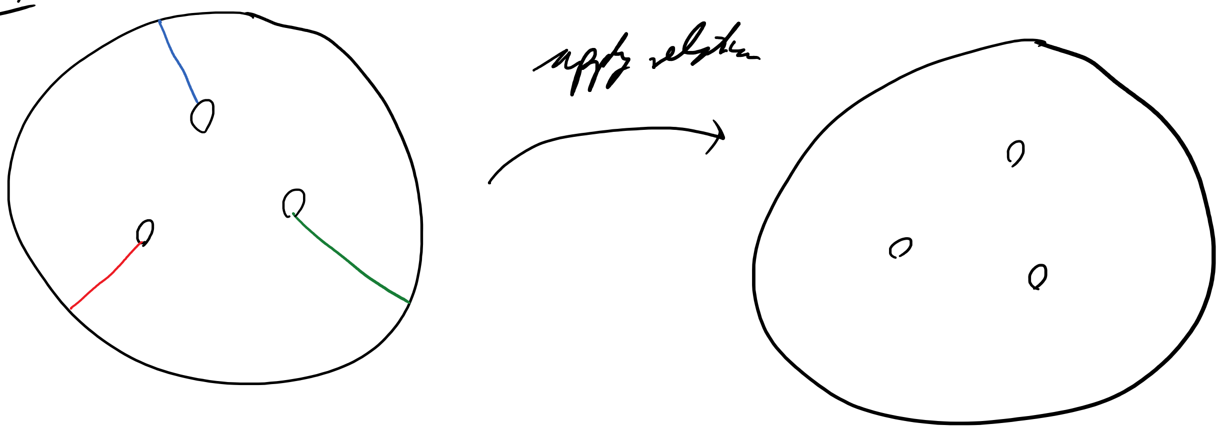
$\Rightarrow \phi(\mu) \sim \mu$ & $\phi(\lambda) \sim \lambda$ $\quad (\phi_*(\mu) = \mu$ & $\phi_*(\lambda) = \lambda)$

After isotopy $\phi(\mu) = \mu$ & $\phi(\lambda) = \lambda$

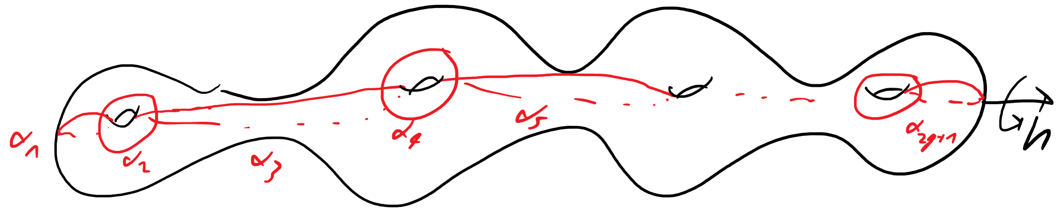


CUT along μ & λ:
get a homeo $D^2 \xrightarrow{\cong} D^2$ fixing ∂D^2
isotopic to id. by Alex. trick.
$\Rightarrow \phi \sim id$ ∎

Ex 4:

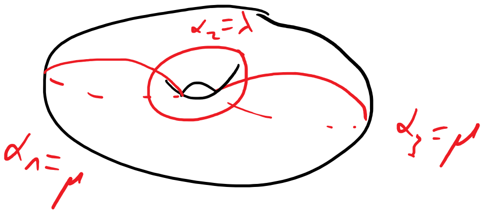


Bonus:

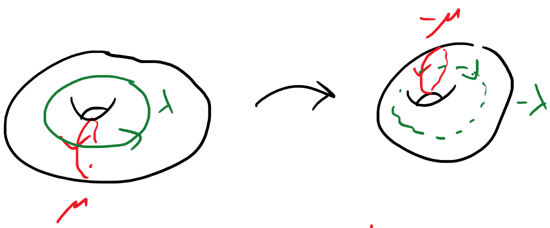


$$\boxed{\text{CLAIM: } [h] = T_{\alpha_{2g+1}} T_{\alpha_{2g}} \cdots T_{\alpha_1} T_{\alpha_1} \cdots T_{\alpha_{2g+1}}}$$

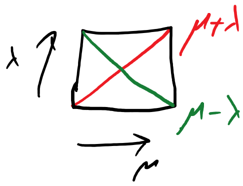
CASE: $g=1$



$$[h] = (T_\mu T_\lambda T_\mu)^2$$



$$h = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$T_\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad T_\lambda = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$T_\mu T_\lambda T_\mu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrel{\hat{=}} \text{rot by } 90°$$

SHEET 6:

Exercise 1.

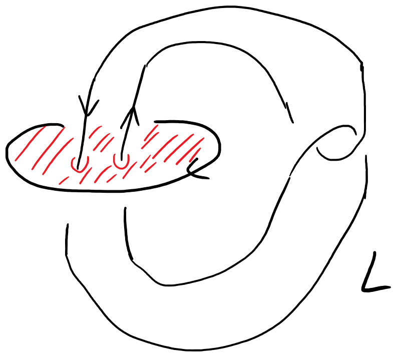
(a) Construct two linked oriented knots with vanishing linking numbers.

(b) Let K_1 and K_2 oriented knots in S^3. Let Σ_2 be a Seifert surface of K_2, see the bonus exercise from Sheet 2. Then the linking number of K_1 and K_2 can be computed as

$$\operatorname{lk}(K_1, K_2) = K_1 \bullet \Sigma_2$$

where $K_1 \bullet \Sigma_2$ denotes the oriented count of transverse intersections of K_1 and Σ_2.

(a)



WHITEHEAD LINK:

$\operatorname{lk} = 0$

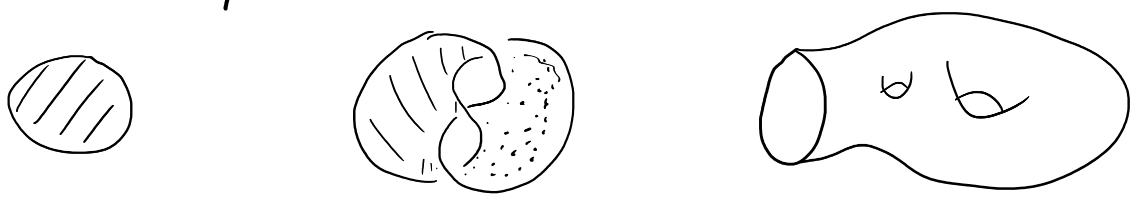
$L \sim \bigcirc\bigcirc$

comp. $= V(L) \neq V(\bigcirc\bigcirc) = -(q^{-1/2} + q^{1/2})$

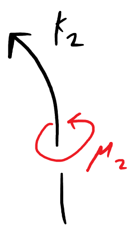
(b) Recall: (SZ BE)

Let $K \subset S^3$ be an or. knot.

$F^2 \subset S^3$ comp. or. is called SEIFERT SURFACE $\Leftrightarrow \partial F = K$

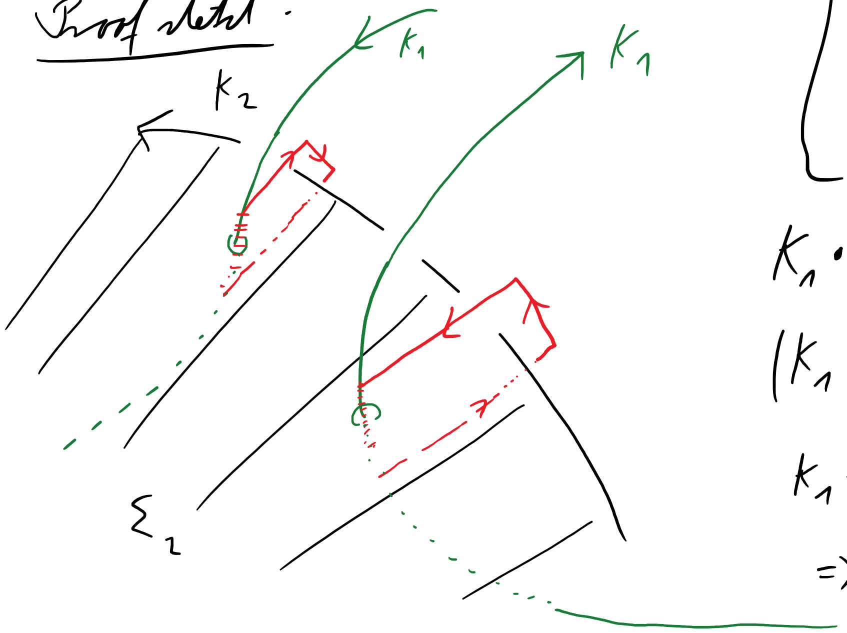


CLAIM: $\operatorname{lk}(K_1, K_2) = K_1 \bullet \Sigma_2$ $\qquad \partial \Sigma_2 = K_2$



$\left[\operatorname{lk}(\pm\mu_2, K_2) = \pm 1 \right]$

Proof sketch:



$K_1 \bullet \Sigma_2 =: n \in \mathbb{Z}$

$(K_1 + n\mu_2) \bullet \Sigma_2 = 0$

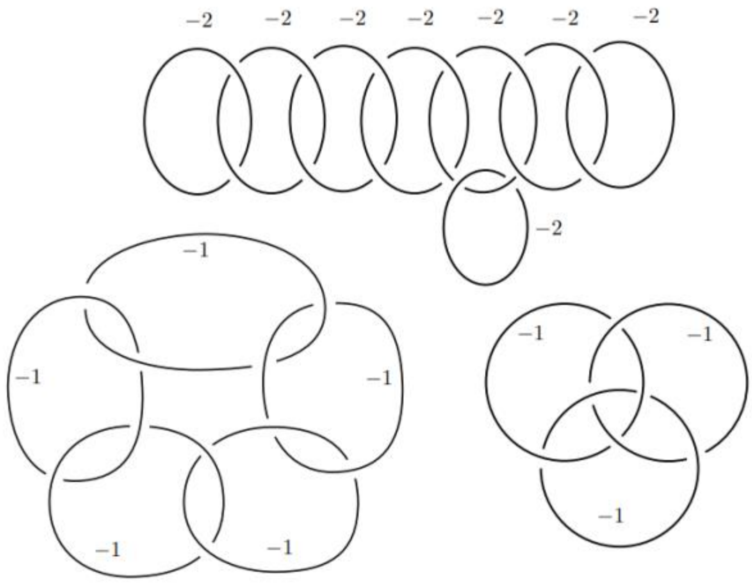
$K_1 + n\mu_2$ & K_2 are unlinked

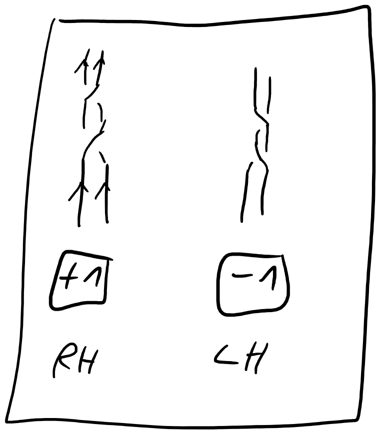
$\Rightarrow \operatorname{lk}(K_1, K_2) = K_1 \bullet \Sigma_2$ $\blacksquare$

Exercise 2.

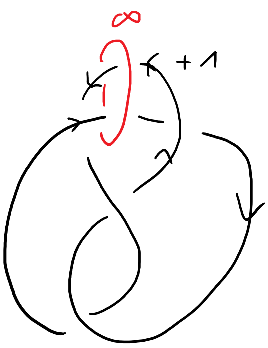
(a) (-1)-surgery along the right-handed trefoil yields the same manifold as $(+1)$-surgery along the figure eight.

(b) Show that all three surgery descriptions in Figure 1 represent the Poincaré homology sphere.





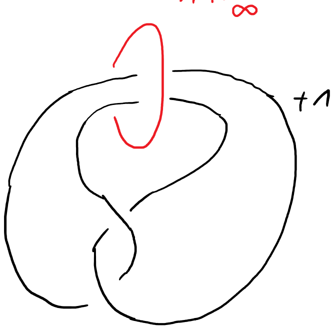
(a)



(-1)-~~fold~~RT

=

$\frac{1}{-1+\frac{1}{\infty}} = -1$



ISOTOPY INTERCHANGING THE COMP

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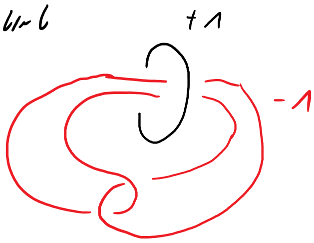
REIDEMEISTER-MOVES



(-1)-~~fold~~RT

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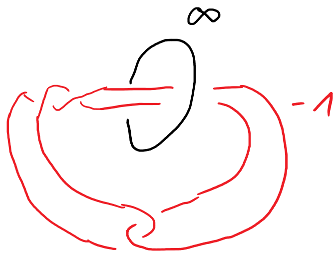

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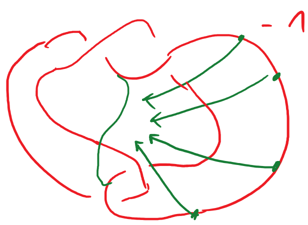

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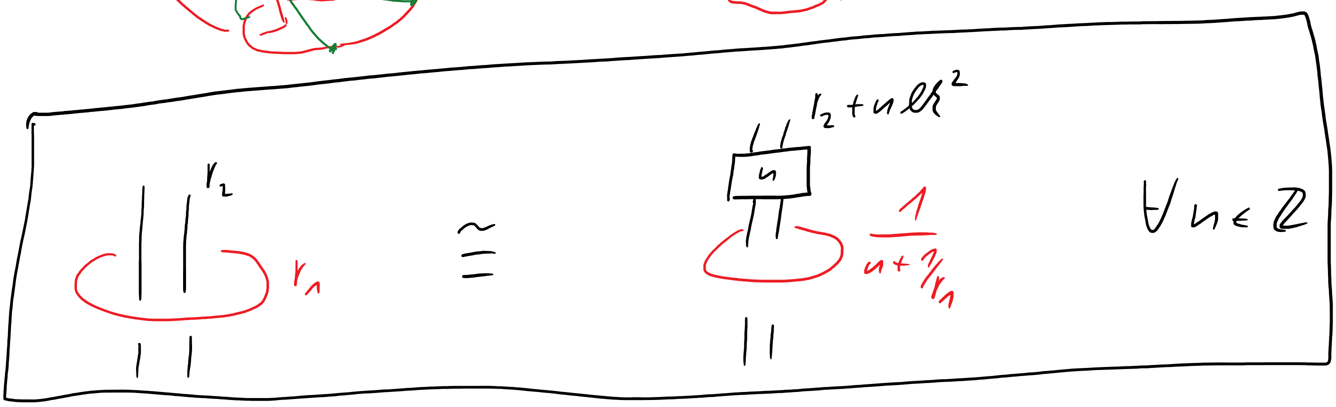
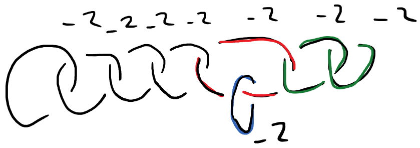
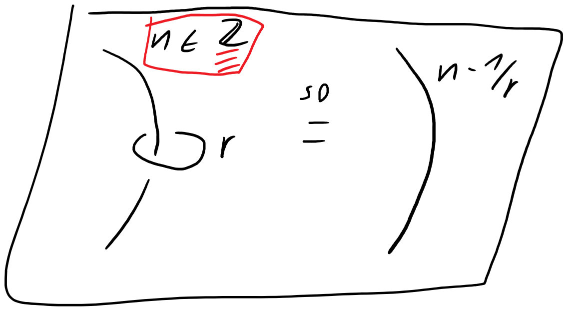
right-handed trefoil



(b)





SLAM DUNKS

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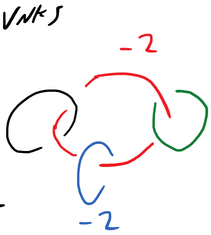


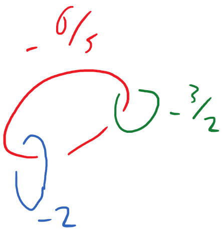
$-2 + 1/2 = -\frac{3}{2}$

$$-2 - \cfrac{1}{-2 - \cfrac{1}{-2 - \cfrac{1}{-2}}}$$

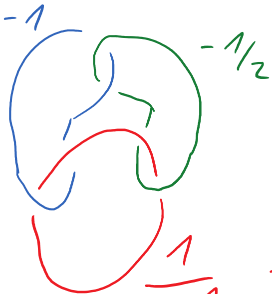
$$\|$$

$$-\frac{5}{4}$$

SD =

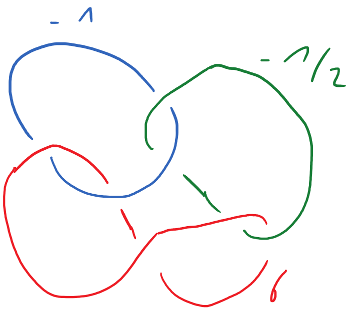


(+1) RT =

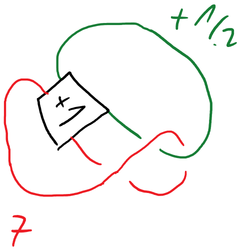


$$\frac{1}{1 + \frac{1}{-\frac{6}{5}}} = 6$$

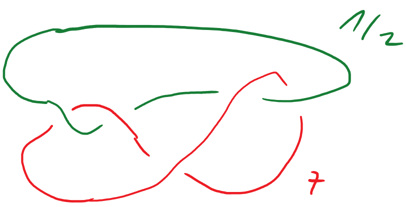
ISOTOPY =



(+1) RT =

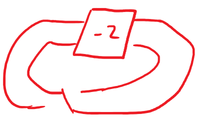


Iso =



Iso =

(-2) RT =



$7 - 2 \cdot \underbrace{lk^2}_{=4} = -1$

=

= P

(the others similar)

Exercise 3.

(a) The lens spaces $L(p, q)$ and $L(p, q + np)$ are homeomorphic for every integer $n \in \mathbb{Z}$.

(b) If $qq' \equiv 1 \bmod(p)$, then the lens spaces $L(p, q)$ and $L(p, q')$ are homeomorphic.

(c) Moreover, are $L(-p, q)$, $L(p, -q)$ and $-L(p, q)$ orientation preserving homeomorphic.

Remark: The relations from (a), (b) and (c) give the complete classification of lens spaces up to orientation preserving homeomorphisms. However, the classification of lens spaces up to homotopy equivalence differs. Two lens spaces $L(p, q)$ and $L(p, q')$ are orientation preserving homotopy equivalent if and only if qq' is a square mod(p). For example $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent but not homeomorphic.

(d) (+5)-surgery along the right-handed trefoil yields yields a lens space.

(e) Describe a surgery presentation of the connected sum of any two lens spaces.

(f) (+6)-surgery along the right-handed trefoil yields the connected sum of two lens spaces.

(a) $L(p,q) = \bigcirc^{-p/q} \overset{(-n)\text{-fold RT}}{=} \bigcirc^{\frac{1}{-n+\frac{1}{-p/q}}} = \bigcirc^{\frac{p}{-np-q}}$

$= \bigcirc^{-\frac{p}{q+np}} = L(p, q+np) \quad \forall n \in \mathbb{Z}$

(b) $L(p,q) = V_1 \cup_\varphi V_2$

$\mu_1 \longmapsto q\mu_2 - p\lambda_2$

$\lambda_1 \longmapsto s\mu_2 + r\lambda_2$

where $qr + ps = -1$ (*)

$= V_2 \cup_{\varphi^{-1}} V_1$

$\mu_2 \longmapsto -r\mu_1 + p\lambda_1$

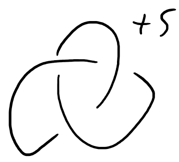
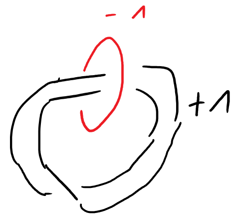
$\lambda_2 \longmapsto -s\mu_1 - q\lambda_1$

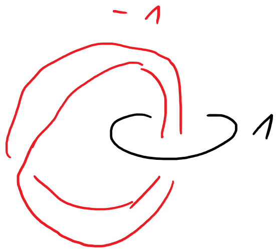
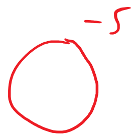
$= L(-p, -r)$

and $qr \equiv -1 \mod(p)$ by (*)

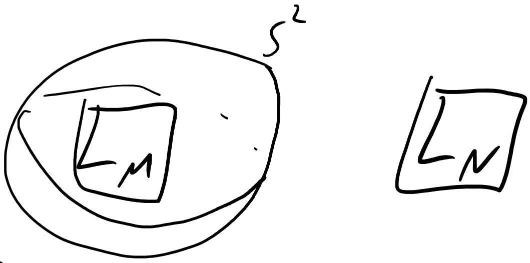
maybe sign mistake?

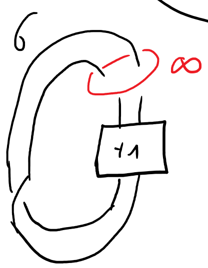
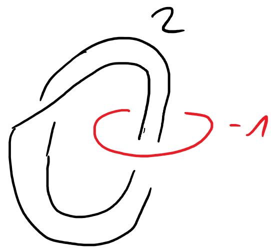
(c) $L(-p,q) = \bigcirc^{p/q} = L(p,-q) = -L(p,q)$

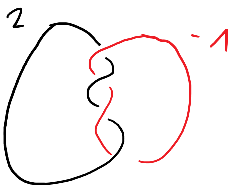
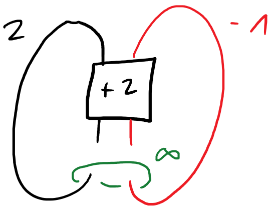
(d)  =  $\overset{(-1)\,RT}{=}$  $= 5 - 1\cdot 2^2$

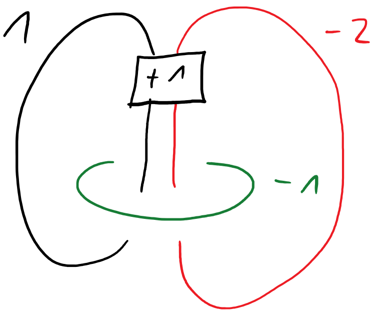
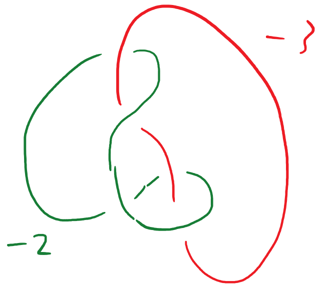
$\overset{\text{ISOTOPY}}{=}$  $\overset{(-1)\,RT}{=}$  =  $= L(5,1)$

(e) $L(p,q) \# L(p',q') =$ 

$M \# N =$ 

(f)  =  $\overset{(-1)\,RT}{=}$ 

$\overset{\text{ISOTOPY}}{=}$  = 

$\overset{(-1)\,RT}{=}$  $\overset{(+1)\,RT}{=}$ 

= 

$= L(2,1) \# L(3,1)$ ∎

SURGERY NUMBER:

$s(M) := \min \{ \#|L| \mid M \text{ is surgery along } L \subset S^3 \}$

$s(M_1) + s(M_2) \geq s(M_1 \# M_2)$ c.f. Heeg. genus

Exercise 4.

(a) Compute the homology groups of a 3-manifold from one of its surgery presentations, i.e. prove Lemma 5.8 from the lecture.

(b) Show that, we cannot get the 3-torus T^3 by surgery along a link with less than 3 components. Describe a surgery diagram of the 3-torus along a 3-component link.

(c) For every natural number $k \in \mathbb{N}$ there exists a 3-manifold that can be obtained by surgery along k-component link but not along a link with less than k components.

(a) $M = S^3_L(r_1, \dots, r_n) \qquad r_i = p_i/q_i$

$$\Rightarrow H_1(M) = \langle \mu_1, \dots, \mu_n \mid p_i\mu_i + q_i \sum_{i \neq j} lk(L_i, L_j)\,\mu_j = 0 \rangle_{\mathbb{Z}}$$

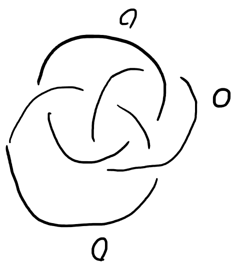
OBSERVATION: $rk(H_1(M)) \leq |L|$

(c) $*\ rk(\#_k S^1\times S^2) = k$ | $*\ \#_k S^1\times S^2 = \underbrace{O^0 \ \dots \ O^0}_{k\text{-times}}$

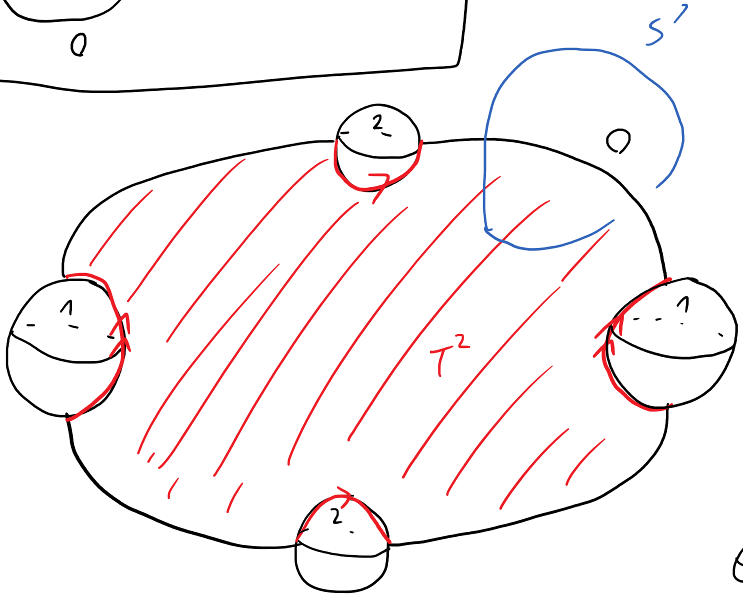
$\Rightarrow s(\#_k S^1\times S^2) \geq k$ | $\Rightarrow s(\#_k S^1\times S^2) \leq k$

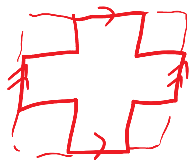
(b) $H_1(T^3) = \mathbb{Z}^3 \quad \Rightarrow \quad s(T^3) \geq 3$

Find a s.d. along 3-comp link of T^3



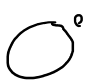
CLAIM $= T^3 \qquad \Rightarrow s(T^3) = 3$





$= T^3$

$= S^1 \times T^2$

 $= S^1\times S^2 =$ 

(a) $L = L_1, \dots, L_n$

$$H_1(S^3 \setminus \mathring{V}L) \cong \mathbb{Z}^n\langle \mu_1, \dots, \mu_n \rangle$$

$$\begin{tikzcd}
\partial VL \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^3 \setminus VL \arrow[d] \\
VL \arrow[r] & S^3
\end{tikzcd}$$

MVT:

$$0 = H_2(S^3) \longrightarrow H_1(\partial VL) \xrightarrow{\cong} H_1(VL) \oplus H_1(S^3 \setminus VL) \longrightarrow H_1(S^3) = 0$$

$$H_1(\partial VL) = H_1\Big(\bigsqcup_{i=1}^n S^1 \times S^1\Big) \cong \mathbb{Z}^{2n}\langle \mu_i, \lambda_i \rangle \qquad H_1(VL) = H_1\Big(\bigsqcup_{i=1}^n S^1 \times D^2\Big) \cong \mathbb{Z}^n\langle \lambda_i \rangle$$

$$\Rightarrow \quad \mathbb{Z}^n\langle \mu_i \rangle$$

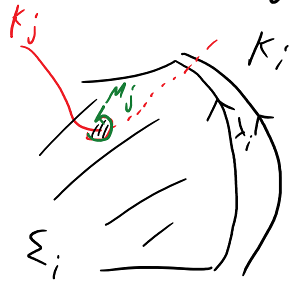
$$\bigsqcup_{i=1}^n S^1 \times D^2 \quad \cup \quad S^3 \setminus VL$$

$$\mu_0^i \longmapsto p_i \mu_i + q_i \lambda_i$$

$$H_1(M) = \langle \mu_i \mid p_i \mu_i + q_i \lambda_i \rangle_{\mathbb{Z}}$$

TO SHOW: $\lambda_i = \sum_{j \neq i} lk(K_i, K_j)\mu_j$

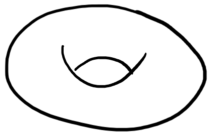
$\lambda_i = \partial \Sigma_i$



$$\lambda_i - \sum_{j \neq i} lk(K_i, K_j)\mu_j = \partial \Sigma \setminus D_i^2 = 0 \in H_1$$

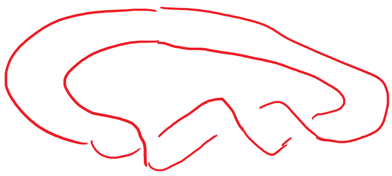
Σ_i = Seifert surface of K_i

BLATT 7 A2 $p, q \in \mathbb{Z}$

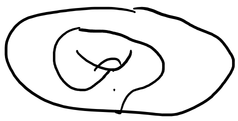
(d) $T_{p,q}$ = TORUS-LINK = $p\mu + q\lambda$ auf $\partial\Big($  $\Big) \subset S^3$

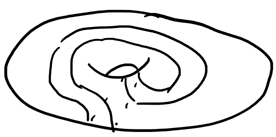
$T_{3,2} = 3\mu + 2\lambda$

||



= trefoil

$T_{1,1} =$  $=$ 

$T_{2,2} =$  $=$ 

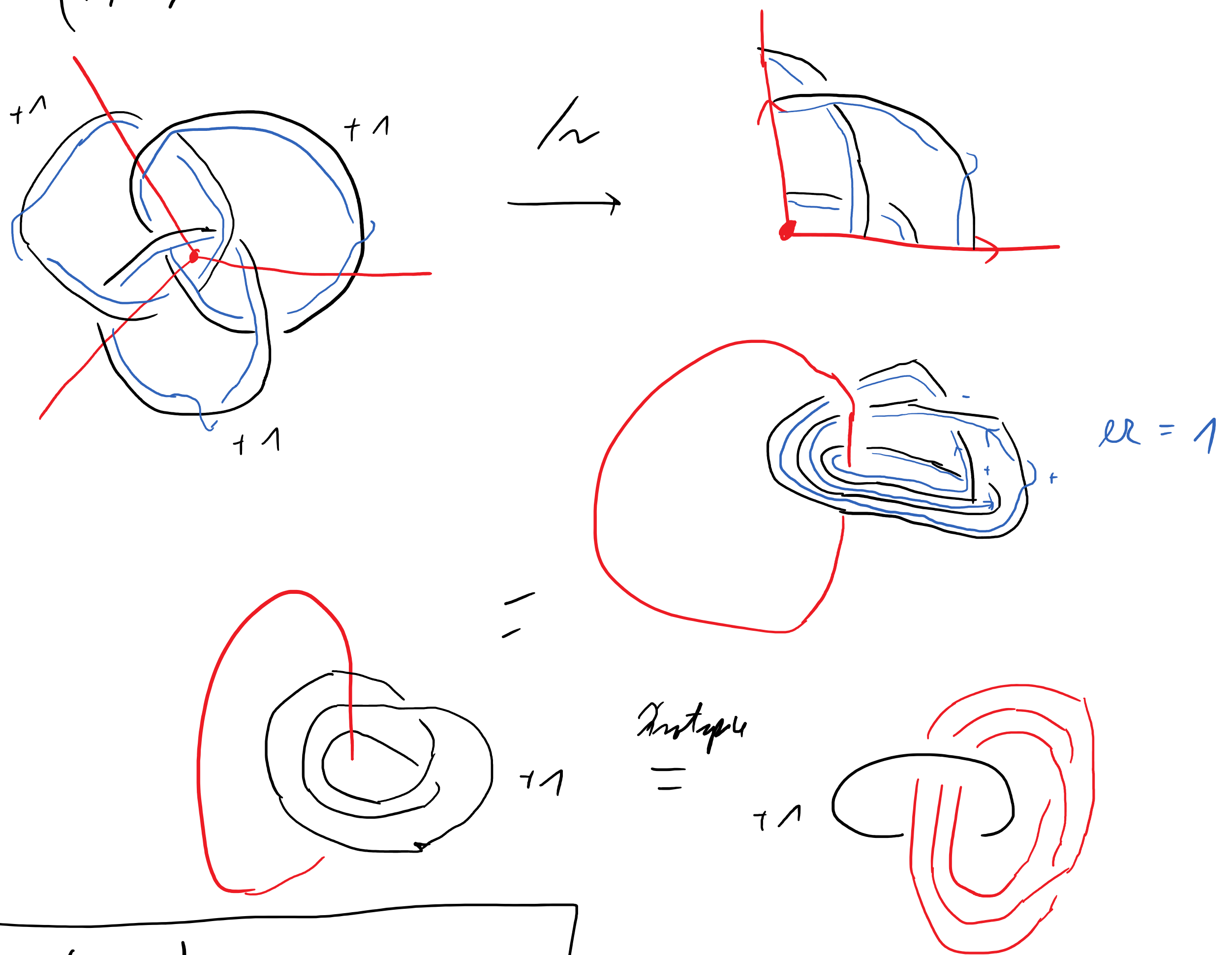
$T_{p,q} = T_{q,p}$

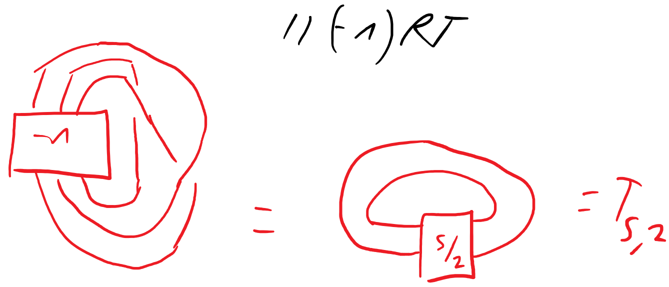
(b) Bsp. $\forall$ Permutation (p,q,r) von $(5,2,3)$

$\exists\ P \longrightarrow S^3$ p-fach verzw. Überl. verzweigt entlang $T_{r,s}$

Bew: * $T_{2,3}$ = trefoil $\quad (p,q,r) = (5,2,3)$ mit T. 6.8 (2)

* $(p,q,r) = (3,2,5)$





* $(p,q,r) = (2,3,5)$